J. Indones. Math. Soc.
Vol. 32, No. 03 (2026), pp. 1–14.

# Characteristic Polynomial and Energy of Matrices Associated with Power Graphs of Cyclic Groups

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**Abstract.** Algebraic graph theory is a fast-expanding field with numerous applications, particularly in the study of power graphs of finite groups. This study focuses on the power graph of cyclic groups with prime power order. Several key results are presented, including the characteristic polynomial and the energies of different matrices associated with the power graph. These matrices include the degree sum, degree exponent, degree subtraction, maximum degree, and degree square sum matrix. The research establishes characteristic polynomial for each matrix and computes their respective energies, revealing the complex relationships between the structure of the graph and its algebraic properties. This work extends the understanding of power graphs and contributes valuable insights into their algebraic and spectral properties.

*energies, power graph, cyclic group, chemical topological graph*

## 1. INTRODUCTION

Knowing the $\pi$-electron energy of conjugated carbon molecules in chemistry that had figured out using Hückel theory, aligns perfectly with the concept of energy as defined in the realm of graph theory. This remarkable coincidence provides a special significance to results related to graph energy. The Hückel theory in quantum chemistry provides a method to calculate the energy levels of $\pi$-electrons in conjugated carbon molecules [1]. When these energy calculations are paralleled with graph energy, a fascinating intersection of chemistry and mathematics emerges.

Similarly, in the study of algebraic structures, graph theory offers a powerful framework for understanding the properties of groups [2]. The power graph, for instance, provides valuable insights into the algebraic structure of the group [3] or

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*2020 Mathematics Subject Classification*: 05C25.
Received: 09-07-2025, accepted: 08-02-2026.

on the ring [4]. It reflects the order of elements, and subgroup relationships, and can highlight symmetries and other structural properties. Power graphs are often simpler than other associated graphs, such as the Cayley or commuting graph, making them useful for studying finite groups. The power graph of a group $G$ is a graph whose vertex set consists of the elements of the group, and two vertices are adjacent if one is a power of the other [5]. Building on this, we derive the general formulas of various graph energies for the power graph on cyclic group. Some scholars have examined the intersection power graph of groups, determining that the graph is Eulerian when the group's order is even [6]. Others have contributed findings on the rainbow connection number for the power graph of finite groups [7]. Additionally, researchers have identified the automorphism groups for various power graphs of finite groups [8], and recently, there has been a growing interest in the study of graph energy for the non-coprime graph associated with the dihedral group[9] and on integer modulo group [10].

Recently, Putra et. al. have published the Sombor Energy of the nilpotent graph of the integer modulo group, which is also one of the cyclic groups [11]. In this article, we investigate not just Sombor energy but including degree sum energy, degree exponential energy, degree subtraction energy, maximum degree adjacency energy, degree square sum energy, Sombor energy, and Randić energy on the cyclic group in general, especially for the prime power order. We also identified a correlation between the graph spectrum and the corresponding energy values obtained.

## 2. Energy of Graph

In this section, we aim to calculate the characteristic polynomial and graph energy of the power graph of cyclic groups with prime power order. These calculations include degree sum energy, degree exponential energy, degree subtraction energy, maximum degree adjacency energy, degree square sum energy, Sombor energy, and Randić energy. Below is the definition of the energy of a graph.

**Definition 2.1.** [1] *Let $\Gamma$ be a graph and $\eta$ be an eigenvalue of the matrix graph of $\Gamma$, then the energy of $\Gamma$ is defined as*

$$E(\Gamma) = \sum |\eta|. \tag{1}$$

Finding the determinant of a small matrix is very straightforward, but not for a large matrix size. This determinant is very important for finding the eigenvalues of a matrix. Below is a property that can help us to find the determinant of a square matrix of order $\sigma$.

**Lemma 2.2.** [12] *If we have a square matrix with order $\sigma$, where $\mu$ and $\nu$ are real numbers, then*

$$\begin{vmatrix} \mu & \nu & \cdots & \nu \\ \nu & \mu & \cdots & \nu \\ \vdots & \vdots & \ddots & \vdots \\ \nu & \nu & \cdots & \mu \end{vmatrix} = (\mu-\nu)^{\sigma-1}[\mu+(\sigma-1)\nu].$$

2.1. **Degree Sum Energy.**

In this section, we explore the concept of degree sum energy, examining its definition, properties, and significance within the context of our study. We analyze how degree sum energy is calculated, its implications for the power graph, and its role in understanding the underlying algebraic structure.

**Definition 2.3.** [13] *Let $\Gamma$ be a graph with $\mathcal{E}$ as a set of its edges, then the matrix of degree sum of $\Gamma$ is defined as $DS(\Gamma) = [ds_{ij}]$ with,*

$$ds_{ij} = \begin{cases} d_i + d_j & , \text{ if } v_i v_j \in \mathcal{E} \\ 0 & , \text{ else.} \end{cases} \tag{2}$$

*where $d_i, d_j$ are the degree of vertices $v_i$ and $v_j$, respectively.*

In the following theorem, we calculate the characteristic polynomial of the degree-sum matrix for the power graph of a cyclic group of order $\sigma$, denoted as $\Gamma_{C_\sigma}$, where $\sigma$ is a power of prime.

**Theorem 2.4.** *The characteristic polynomial of $DS(\Gamma_{C_\sigma})$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is*

$$P_{DS(\Gamma_{C_\sigma})}(\eta) = [\eta + 2(\sigma-1)]^{\sigma-1}\left[\eta - 2(\sigma-1)^2\right]. \tag{3}$$

*Proof.* Suppose that $C_\sigma = \langle x \rangle$, with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$. For any $a, b \in C_\sigma$ can be written as $a = x^\alpha, b = x^\beta$ with $\alpha, \beta \in \mathbb{Z}$. Thus, we have two cases:

Case 1, Wlog, let $\text{GCD}(\alpha,\sigma) = 1,$and $\text{GCD}(b,\sigma) \neq 1$. So, there exist $k_1, k_2 \in \mathbb{N}$ such that $k_1\alpha + k_2\sigma = 1$ implies $b = x^\beta = x^{\beta(k_1\alpha + k_2\sigma)} = x^{\beta k_1 \alpha} \cdot x^{\beta k_2 \sigma} = (x^\alpha)^{k_1\beta} \cdot (x^\sigma)^{k_2\beta} = a^{k_1\beta} \cdot e = a^{k_1\beta}$. Thus, $b$ is a power of $a$.

Case 2, If $\alpha, \beta$ and $\sigma$ are not relatively prime, then $\alpha = p^l, \beta = p^m$, for some $k \geq l, m \in \mathbb{Z}$. Wlog, let $l < m$, so there exists a natural number $q$ such that $m = l+q$. So, $\beta = p^{l+q} = p^l p^q = \alpha p^q$. Then $b = x^\beta = x^{\alpha p^q} = (x^\alpha)^{p^q} = a^{p^q}$. Hence, $a$ and $b$ are adjacent in both cases.

Because all two different elements of $C_\sigma$ are neighbours, the power graphs of $\Gamma_{C_\sigma}$ must be a complete graph with order $\sigma$. So, the degree of all vertices is $\sigma - 1$.

Then by Definition 2.3 we get the degree sum matrix of $C_\sigma$ is

$$DS(\Gamma_{C_\sigma}) = \begin{pmatrix} 0 & 2\sigma-2 & \cdots & 2\sigma-2 \\ 2\sigma-2 & 0 & \cdots & 2\sigma-2 \\ \vdots & \vdots & \ddots & \vdots \\ 2\sigma-2 & 2\sigma-2 & \cdots & 0 \end{pmatrix}.$$

Therefore, the characteristic polynomial of $\Gamma_{C_\sigma}$ is found by calculating $|\eta I - DS(\Gamma_{C_\sigma})|$ which gives

$$P_{DS(\Gamma_{C_\sigma})}(\eta) = \begin{vmatrix} \eta & -(2\sigma-2) & \cdots & -(2\sigma-2) \\ -(2\sigma-2) & \eta & \cdots & -(2\sigma-2) \\ \vdots & \vdots & \ddots & \vdots \\ -(2\sigma-2) & -(2\sigma-2) & \cdots & \eta \end{vmatrix}.$$

By using Lemma 2.2, we get

$$\begin{aligned} P_{DS(\Gamma_{C_\sigma})}(\eta) =& [\eta - (-(2\sigma-2))]^{\sigma-1} [\eta + (\sigma-1)(-(2\sigma-2))] \\ =&(\eta + 2(\sigma-1))^{\sigma-1}(\eta - 2(\sigma-1)^2). \end{aligned}$$

□

**Theorem 2.5.** *The degree sum energy of* $\Gamma_{C_\sigma}$ *with* $\sigma = p^k$ *where* $p$ *is a prime number and* $k \in \mathbb{N}$ *is* $4(\sigma-1)^2$.

*Proof.* By Theorem 2.4, we obtain the eigenvalues of the degree sum matrix of $\Gamma_{C_\sigma}$ by solving $P_{DS(\Gamma_{C_\sigma})}(\eta) = 0$ which gives

$$(\eta + 2(\sigma-1))^{\sigma-1} = 0 \text{ or } \eta + 2(\sigma-1)^2 = 0.$$

Thus, $\eta = -2(\sigma-1)$ with multiplicity $\sigma-1$ and $\eta = 2(\sigma-1)^2$ with multiplicity 1. Using Definition 2.1 which states that the degree sum energy of $\Gamma_{C_\sigma}$ is the sum of all its absolute eigenvalues, we find that the degree sum energy of $\Gamma_{C_\sigma}$ is

$$\begin{aligned} E(DS(\Gamma_{C_\sigma})) =& \sum |\eta| \\ =&(\sigma-1)| - 2(\sigma-1)| + |2(\sigma-1)^2| \\ =&2(\sigma-1)^2 + 2(\sigma-1)^2 \\ =&4(\sigma-1)^2. \end{aligned}$$

□

2.2. **Degree Exponent Energy.**

In this section, we explore the concept of the characteristic polynomial and the degree exponent energy in greater detail. We examine how the characteristic polynomial is derived and its properties. Additionally, we discuss the degree exponent energy, focusing on its significance and its role in understanding the behavior of certain functions or systems.

**Definition 2.6.** [14] *Let $\Gamma$ be a graph, then the matrix of degree exponent of $\Gamma$ is $DE(\Gamma) = [de_{ij}]$ with*

$$de_{ij} = \begin{cases} d_i^{d_j} & , \text{ if } i \neq j \\ 0 & , \text{ if } i = j. \end{cases} \tag{4}$$

*where $d_i, d_j$ are the degree of vertices $v_i$ and $v_j$, respectively.*

**Theorem 2.7.** *The characteristic polynomial of $DE(\Gamma_{C_\sigma})$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is*

$$P_{DE(\Gamma_{C_\sigma})}(\eta) = (\eta + (\sigma-1)^{\sigma-1})^{\sigma-1}(\eta + (\sigma-1)^{\sigma}). \tag{5}$$

*Proof.* Based on the proof of Theorem 2.4, we know that $\forall v \in V(\Gamma_{C_\sigma})$ with $\sigma$ is a prime power has $\sigma - 1$ degrees. By Definition 2.6, we obtain the degree exponent matrix of the graph $\Gamma_{C_\sigma}$ by raising its degrees to the powers of the other vertices' degrees. Thus,

$$DE(\Gamma_{C_\sigma}) = \begin{pmatrix} 0 & (\sigma-1)^{\sigma-1} & \cdots & (\sigma-1)^{\sigma-1} \\ (\sigma-1)^{\sigma-1} & 0 & \cdots & (\sigma-1)^{\sigma-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma-1)^{\sigma-1} & (\sigma-1)^{\sigma-1} & \cdots & 0 \end{pmatrix}.$$

By calculating $|\eta I - DE(\Gamma_{C_\sigma})|$, we obtain the characteristic polynomial of the degree exponent matrix of $\Gamma_{C_\sigma}$, namely

$$\begin{aligned} P_{DE(\Gamma_{C_\sigma})}(\eta) =& (\eta + (\sigma-1)^{\sigma-1})^{\sigma-1}(\eta - (\sigma-1)(\sigma-1)^{\sigma-1}) \\ =& (\eta + (\sigma-1)^{\sigma-1})^{\sigma-1}(\eta - (\sigma-1)^{\sigma}). \end{aligned}$$

□

**Theorem 2.8.** *The degree exponent energy of $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is $2(\sigma-1)^{\sigma}$.*

*Proof.* By solving $P_{DE(\Gamma_{C_\sigma})}(\eta) = 0$ from Theorem 2.7, we obtain

$$\eta = -(\sigma-1)^{\sigma-1} \text{ or } \eta = (\sigma-1)^{\sigma}.$$

The multiplicities are $\sigma - 1$ and 1, respectively. Therefore, the degree exponent energy of $\Gamma_{C_\sigma}$ is

$$\begin{aligned} E(DE(\Gamma_{C_\sigma})) =& (\sigma-1)|-(\sigma-1)^{\sigma-1}| + |(\sigma-1)^{\sigma}| \\ =& (\sigma-1)^{\sigma} + (\sigma-1)^{\sigma} \\ =& 2(\sigma-1)^{\sigma}. \end{aligned}$$

□

## 2.3. **Degree Subtraction Energy.**

This section explores the characteristic polynomial and the degree subtraction energy of the nilpotent graph of $\mathbb{Z}_\sigma$. First, we discuss how the characteristic polynomial is derived and its mathematical properties. The characteristic polynomial helps calculate the eigenvalues of a matrix, which is used to compute the degree of subtraction energy.

**Definition 2.9.** [15] *Let $\Gamma$ be a graph, the degree subtraction matrix of $\Gamma$ is $DSt(\Gamma_{C_\sigma}) = [dst_{ij}]$ with*

$$dst_{ij} = \begin{cases} d_i - d_j & , i \neq j \\ 0 & , i = j. \end{cases} \tag{6}$$

*where $d_i, d_j$ are the degree of vertices $v_i$ and $v_j$, respectively.*

**Theorem 2.10.** *The characteristic polynomial of $DSt(\Gamma_{C_\sigma})$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is*

$$P_{DSt(\Gamma_{C_\sigma})}(\eta) = \eta^\sigma. \tag{7}$$

*Proof.* By Definition 2.9, we can form the degree subtraction matrix of the graph $\Gamma_{C_\sigma}$ with $\sigma$ is a prime power by subtracting the degree of each vertex. Since all vertices of $\Gamma_{C_\sigma}$ have the same degree, $\forall v_i, v_j \in V(\Gamma_{C_\sigma}), \quad d_i = d_j$. Therefore, $dst_{ij} = 0$, and $DSt(\Gamma_{C_\sigma}) = O$, where $O$ is the zero matrix. It is easy to check that $P_{DSt(\Gamma_{C_\sigma})}(\eta) = \eta^\sigma$. □

**Theorem 2.11.** *The degree subtraction energy of $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is 0.*

*Proof.* By Theorem 2.10, the eigenvalues of $DSt(\Gamma_{C_\sigma})$ are all 0. Therefore, the degree subtraction energy of $\Gamma_{C_\sigma}$ is 0. □

## 2.4. **Maximum Degree Adjacency.**

This section explores the characteristic polynomial and the maximum degree energy. The maximum degree matrix is based on the maximum degree of two distinct vertices. The characteristic polynomial is crucial for calculating the eigenvalues of a matrix. Next, we compute the maximum degree energy, which is the summation of absolute eigenvalues of the maximum degree matrix.

**Definition 2.12.** [16] *The maximum degree matrix of graph $\Gamma$ with $\mathcal{E}$ is a set of its edges defined with the maximum degree of vertex of graph $\Gamma$ which is $M(\Gamma) = [m_{ij}]$ with*

$$m_{ij} = \begin{cases} \max(d_i, d_j) & , \text{ if } v_i v_j \in \mathcal{E} \\ 0 & , \text{ else.} \end{cases} \tag{8}$$

**Theorem 2.13.** *The characteristic polynomial of $M(\Gamma_{C_\sigma})$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is*

$$P_{M(\Gamma_{C_\sigma})}(\eta) = [\eta + (\sigma - 1)]^{\sigma - 1}[\eta - (\sigma - 1)^2]. \tag{9}$$

*Proof.* Since $\Gamma_{C_\sigma}$ with $\sigma$ is a prime power is a complete graph, all of its vertices have the same degree. By the proof of Theorem 2.4, the degree of each vertex is $\sigma - 1$. Therefore, $\max(d_i, d_j) = \sigma - 1 \quad \forall v_i, v_j \in V(\Gamma_{C_\sigma})$. Thus, we obtain the maximum degree adjacency matrix of $\Gamma_{C_\sigma}$ below:

$$M(\Gamma_{C_\sigma}) = \begin{pmatrix} 0 & \sigma-1 & \cdots & \sigma-1 \\ \sigma-1 & 0 & \cdots & \sigma-1 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma-1 & \sigma-1 & \cdots & 0 \end{pmatrix}.$$

Thus, the characteristic polynomial of the matrix $M(\Gamma_{C_\sigma})$ is

$$\begin{aligned} P_{M(\Gamma_{C_\sigma})}(\eta) = & \begin{vmatrix} \eta & -(\sigma-1) & \cdots & -(\sigma-1) \\ -(\sigma-1) & \eta & \cdots & -(\sigma-1) \\ \vdots & \vdots & \ddots & \vdots \\ -(\sigma-1) & -(\sigma-1) & \cdots & \eta \end{vmatrix} \\ =&(\eta - (-(\sigma-1)))^{\sigma-1}(\eta + (\sigma-1)(-(\sigma-1))) \\ =&(\eta + (\sigma-1))^{\sigma-1}(\eta - (\sigma-1)^2). \end{aligned}$$

□

**Theorem 2.14.** *The maximum degree energy of graph $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is $2(\sigma-1)^2$.*

*Proof.* By Theorem 2.13, the eigenvalues of the matrix $M(\Gamma_{C_\sigma})$ are $-(\sigma-1)$ with multiplicity $\sigma-1$ and $(\sigma-1)^2$ with multiplicity 1. Therefore, by the definition of energy, the energy of the maximum degree matrix of $\Gamma_{C_\sigma}$ is

$$\begin{aligned} E(M(\Gamma_{C_\sigma})) =&(\sigma-1)|-(\sigma-1)| + |(\sigma-1)^2| \\ =&(\sigma-1)^2 + (\sigma-1)^2 \\ =&2(\sigma-1)^2. \end{aligned}$$

□

2.5. **Degree Square Sum.**

The degree square sum matrix was defined similarly to the degree sum matrix. However, before we summed the vertex's degrees, each degree value was squared. By calculating its characteristic polynomial, we determine its eigenvalues to compute the degree square sum energy of the nilpotent graph of $\Gamma_{C_\sigma}$.

**Definition 2.15.** [17] *The degree square sum matrix of graph $\Gamma$ is defined as $DSS(\Gamma) = [dss_{ij}]$ with*

$$dss_{ij} = \begin{cases} d_i^2 + d_j^2 & , i \neq j \\ 0 & , \textit{ else.} \end{cases} \tag{10}$$

**Theorem 2.16.** *The characteristic polynomial of $DSS(\Gamma_{C_\sigma})$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is*

$$P_{DSS(\Gamma_{C_\sigma})}(\eta) = [\eta + 2(sigma - 1)^2]^{\sigma-1}[\eta - 2(\sigma - 1)^3]. \tag{11}$$

*Proof.* By the definition of the degree square sum matrix, we can form the matrix by adding the squares of the degrees of two distinct vertices of $\Gamma_{C_\sigma}$ with $\sigma$ is a prime power. Thus,

$$DSS(\Gamma_{C_\sigma}) = \begin{pmatrix} 0 & 2(\sigma-1)^2 & \cdots & 2(\sigma-1)^2 \\ 2(\sigma-1)^2 & 0 & \cdots & 2(\sigma-1)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 2(\sigma-1)^2 & 2(\sigma-1)^2 & \cdots & 0 \end{pmatrix}.$$

Thus, the characteristic polynomial of matrix $DSS(\Gamma_{C_\sigma})$ is

$$\begin{aligned} P_{DSS(\Gamma_{C_\sigma})}(\eta) = &\begin{vmatrix} \eta & -(2(\sigma-1)^2) & \cdots & -(2(\sigma-1)^2) \\ -(2(\sigma-1)^2) & \eta & \cdots & -(2(\sigma-1)^2) \\ \vdots & \vdots & \ddots & \vdots \\ -(2(\sigma-1)^2) & -(2(\sigma-1)^2) & \cdots & \eta \end{vmatrix} \\ =&(\eta - (-(2(\sigma-1)^2)))^{\sigma-1}(\eta + (\sigma-1)(-2(\sigma-1)^2)) \\ =&(\eta + 2(\sigma-1)^2)^{\sigma-1}(\eta - 2(\sigma-1)^3). \end{aligned}$$

□

**Theorem 2.17.** *The degree square sum of graph $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is $4(\sigma-1)^3$.*

*Proof.* By Theorem 2.16, the eigenvalues of $DSS(\Gamma_{C_\sigma})$ are $-2(\sigma-1)^2$ with multiplicity $\sigma - 1$ and $2(\sigma-1)^3$ with multiplicity 1. Therefore, the degree square sum energy of $\Gamma_{C_\sigma}$ is

$$\begin{aligned} E(DSS(\Gamma_{C_\sigma})) =&(\sigma-1)|-2(\sigma-1)^2| + |2(\sigma-1)^3| \\ =&2(\sigma-1)^3 + 2(\sigma-1)^3 \\ =&4(\sigma-1)^3. \end{aligned}$$

□

### 2.6. **Sombor Energy.**

The Sombor energy is the summation of absolute eigenvalues of the Sombor matrix. In this section, we examine the characteristic polynomial to get the eigenvalues of the Sombor matrix to calculate its Sombor energy. We begin with the Sombor matrix, defined as the square root of the sum of the squares of the degrees of two adjacent vertices on the graph.

**Definition 2.18.** [18] *Let $\Gamma$ be a graph, $SO(\Gamma) = [so_{ij}]$ defined Sombor matrix of graph $\Gamma$ with,*

$$so_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & , \text{ if } v_i v_j \in \mathcal{E}(G) \\ 0 & , \text{ else.} \end{cases} \tag{12}$$

**Theorem 2.19.** *Let $\Gamma_{C_\sigma}$ is power graph of $C_\sigma$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$. Then the characteristic polynomial of $SO(\Gamma_{C_\sigma})$ is*

$$P_{SO(\Gamma_{C_\sigma})}(\eta) = [\eta + (\sigma - 1)\sqrt{2}]^{\sigma-1}[\eta - (\sigma - 1)^2\sqrt{2}]. \tag{13}$$

*Proof.* Based on Definition 2.18, we obtain the Sombor matrix of $\Gamma_{C_\sigma}$ with $\sigma$ is a prime power, which is

$$SO(\Gamma_{C_\sigma}) = \begin{pmatrix} 0 & (\sigma-1)\sqrt{2} & \cdots & (\sigma-1)\sqrt{2} \\ (\sigma-1)\sqrt{2} & 0 & \cdots & (\sigma-1)\sqrt{2} \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma-1)\sqrt{2} & (\sigma-1)\sqrt{2} & \cdots & 0 \end{pmatrix}.$$

Thus, the characteristic polynomial of $SO(\Gamma_{C_\sigma})$ is

$$\begin{aligned} P_{SO(\Gamma_{C_\sigma})}(\eta) = & \begin{vmatrix} \eta & -(\sigma-1)\sqrt{2} & \cdots & -(\sigma-1)\sqrt{2} \\ -(\sigma-1)\sqrt{2} & \eta & \cdots & -(\sigma-1)\sqrt{2} \\ \vdots & \vdots & \ddots & \vdots \\ -(\sigma-1)\sqrt{2} & -(\sigma-1)\sqrt{2} & \cdots & \eta \end{vmatrix} \\ = & (\eta - (-(\sigma - 1\sqrt{2})))^{\sigma-1}(\eta + (\sigma-1)(-(\sigma-1)\sqrt{2})) \\ = & (\eta + (\sigma-1)\sqrt{2})^{\sigma-1}(\eta - (\sigma-1)^2\sqrt{2}). \end{aligned}$$

□

**Theorem 2.20.** *The Sombor energy of $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is equal to $2\sqrt{2}(\sigma-1)^2$.*

*Proof.* By Theorem 2.19, the eigenvalues of $SO(\Gamma_{C_\sigma})$ are $-(\sigma-1)\sqrt{2}$ with multiplicity $\sigma - 1$ and $(\sigma-1)^2\sqrt{2}$ with multiplicity 1. Therefore, we obtain

$$\begin{aligned} E(SO(\Gamma_{C_\sigma})) = & (\sigma-1)|-(\sigma-1)\sqrt{2}| + |(\sigma-1)^2\sqrt{2}| \\ = & (\sigma-1)^2\sqrt{2} + (\sigma-1)^2\sqrt{2} \\ = & 2\sqrt{2}(\sigma-1)^2. \end{aligned}$$

□

2.7. **Randić Energy.**

Randić matrix is defined as the reciprocal of the square root of the product of two degrees. In this section, we calculate the characteristic polynomial and Randić energy by summing the absolute eigenvalues of the Randić matrix.

**Definition 2.21.** [19] *Let $\Gamma$ be a graph, $R(\Gamma) = [r_{ij}]$ defined Randić matrix of graph $\Gamma$ with,*

$$r_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i d_j}} & , \text{ if } v_i v_j \in \mathcal{E}(\Gamma) \\ 0 & , \text{ else.} \end{cases} \tag{14}$$

**Theorem 2.22.** *The characteristic polynomial of $R(\Gamma_{C_\sigma})$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is*

$$P_{R(\Gamma_{C_\sigma})}(\eta) = [\eta + (\sigma - 1)^{-2}][\eta - (\sigma - 1)^{-1}]. \tag{15}$$

*Proof.* Based on Definition 2.21 and the proof of Theorem 2.4, we can form the Randić energy of $\Gamma_{C_\sigma}$ with $\sigma$ is a prime power, represented by

$$R(\Gamma_{C_\sigma}) = \begin{pmatrix} 0 & \frac{1}{(\sigma-1)} & \cdots & \frac{1}{(\sigma-1)} \\ \frac{1}{(\sigma-1)} & 0 & \cdots & \frac{1}{(\sigma-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(\sigma-1)} & \frac{1}{(\sigma-1)} & \cdots & 0 \end{pmatrix}.$$

Hence, we obtain

$$\begin{aligned} P_{R(\Gamma_{C_\sigma})}(\eta) &= \begin{vmatrix} \eta & -\frac{1}{(\sigma-1)} & \cdots & -\frac{1}{(\sigma-1)} \\ -\frac{1}{(\sigma-1)} & \eta & \cdots & -\frac{1}{(\sigma-1)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{(\sigma-1)} & -\frac{1}{(\sigma-1)} & \cdots & \eta \end{vmatrix} \\ &= \left(\eta - \left(-\left(\frac{1}{(\sigma-1)}\right)\right)\right)^{\sigma-1} \left(\eta + (\sigma - 1)\left(-\frac{1}{(\sigma-1)}\right)\right) \\ &= \left(\eta + \frac{1}{(\sigma-1)}\right)^{\sigma-1} (\eta - 1) \\ &= (\eta + (\sigma-1)^{-1})(\eta - 1). \end{aligned}$$

□

**Theorem 2.23.** *The Randić energy of $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is equal to 2.*

*Proof.* Based on Theorem 2.22, the eigenvalues of $R(\Gamma_{C_\sigma})$ are $-(\sigma-1)^{-2}$ with multiplicity $\sigma-1$ and $(\sigma-1)^{-1}$ with multiplicity 1. Therefore,

$$\begin{aligned} E(R(\Gamma_{C_\sigma})) =&(\sigma-1)|-(\sigma-1)^{-1}|+|1| \\ =&1+1 \\ =&2. \end{aligned}$$

□

Here, we introduce the graph's spectrum and relate it to its energy, showing how spectral characteristics reveal the graph's total energy.

**Definition 2.24.** *Let $\Gamma$ be a graph with $\sigma$ vertices, $\eta_1, \eta_2, \ldots, \eta_\sigma$ be eigenvalues of matrix of graph $\Gamma$ with each multiplicity of its eigenvalues is $\delta_1, \delta_2, \ldots, \delta_\sigma$ then spectral of $\Gamma$ is*

$$spec(\Gamma) = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_\sigma \\ \delta_1 & \delta_2 & \cdots & \delta_\sigma \end{pmatrix}.$$

*Spectral radius of graph $\Gamma$ defined as $\rho_\Gamma = \max(|\eta_i||i = 1, 2, \ldots, \sigma)$.*

From the definition given above and all of the previous results, we obtain the following consequence.

**Corollary 2.25.** *The degree sum energy, degree exponent energy, degree subtraction energy, maximum degree energy, degree square sum energy, Sombor energy and Randić energy of graph $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is two times its spectral radius.*

## 3. Conclusion

Based on Section 2, the characteristic polynomial of the degree sum matrix, degree exponential matrix, degree subtraction matrix, maximum degree adjacency matrix, degree square sum matrix, Sombor matrix, and Randić matrix of nilpotent graph of $\Gamma_{C_\sigma}$ where $\sigma$ is a power of prime number respectively are:

(1) $P_{DS(\Gamma_{C_\sigma})}(\eta) = [\eta + 2(\sigma-1)]^{\sigma-1}\left[\eta - 2(\sigma-1)^2\right]$.
(2) $P_{DE(\Gamma_{C_\sigma})}(\eta) = (\eta + (\sigma-1)^{\sigma-1})^{\sigma-1}(\eta + (\sigma-1)^\sigma)$.
(3) $P_{DSt(\Gamma_{C_\sigma})}(\eta) = \eta^\sigma$.
(4) $P_{M(\Gamma_{C_\sigma})}(\eta) = [\eta + (\sigma-1)]^{\sigma-1}[\eta - (\sigma-1)^2]$.
(5) $P_{DSS(\Gamma_{C_\sigma})}(\eta) = [\eta + 2(\mu-1)^2]^{\sigma-1}[\eta - 2(\sigma-1)^3]$.
(6) $P_{SO(\Gamma_{C_\sigma})}(\eta) = [\eta + (\sigma-1)\sqrt{2}]^{\sigma-1}[\eta - (\sigma-1)^2\sqrt{2}]$.
(7) $P_{R(\Gamma_{C_\sigma})}(\eta) = [\eta + (\sigma-1)^{-2}][\eta - (\sigma-1)^{-1}]$.

From the characteristic polynomial of each matrix, we calculate degree sum energy, degree exponential energy, degree subtraction energy, maximum degree adjacency energy, degree square sum energy, Sombor energy, and Randić energy, respectively namely:

(1) $E(DS(\Gamma_{C_\sigma})) = 4(\sigma - 1)^2$.
(2) $E(DE(\Gamma_{C_\sigma})) = 2(\sigma - 1)^\sigma$.
(3) $E(DSt(\Gamma_{C_\sigma})) = 0$.
(4) $E(M(\Gamma_{C_\sigma})) = 2(\sigma - 1)^2$
(5) $E(DSS(\Gamma_{C_\sigma})) = 4(\sigma - 1)^3$.
(6) $E(SO(\Gamma_{C_\sigma})) = 2\sqrt{2}(\sigma - 1)^2$.
(7) $E(R(\Gamma_{C_\sigma})) = 2$.

The last one is the degree sum energy, degree exponent energy, degree subtraction energy, maximum degree energy, degree square sum energy, Sombor energy and Randić energy of graph $\Gamma_{C_\sigma}$ with $\sigma = p^k$ where $p$ is a prime number and $k \in \mathbb{N}$ is two times its spectral radius.

These results are crucial for initiating research on the energy of the graph, particularly for other cyclic groups or possibly for finite groups.

**Data Availability Statement**
No new data were created or analyzed in this study.

**Declarations**. No conflict of interest.

**Author Contributions**. CRediT (Contributor Roles Taxonomy) Lalu Riski Wirendra Putra: Formal analysis, investigation, project administration, visualization, resources, writing–original draft, writing–review and editing. I Gede Adhitya Wisnu Wardhana: Conceptualization, funding acquisition, methodology, resources, validation, writing–review and editing. Nor Haniza Sarmin: Methodology, supervision, validation, writing–review and editing. All authors discussed the results and contributed to the final manuscript.

**Funding Statement**. This research was financially supported by the 2025 BIMA Fundamental Research – Regular Grant under contract number 079/C3/DT.05.00/PL/2025.

**Acknowledgement.** The authors express their sincere gratitude to the Department of Mathematics at both the University of Mataram (UNRAM) and Universiti Teknologi Malaysia (UTM) for their support and collaboration, which significantly contributed to the completion of this study.

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