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Characterization of $\mathcal{R}(2K_2, F_n)$ with Minimum Order for Small n

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Abstract. A Fan graph F_n is defined as the graph $P_n + K_1$, where P_n is the path on n vertices. The notation $F \rightarrow (G, H)$ means that if all edges of F are arbitrarily colored by red or blue, then either the subgraph of F induced by all red edges contains a graph G or the subgraph of F induced by all blue edges contains a graph H. Let $\mathcal{R}(G, H)$ denote the set of all graphs F satisfying $F \rightarrow (G, H)$ and for every $e \in E(F)$, $(F - e) \not\rightarrow (G, H)$. In this paper, we propose some properties for a graph G of minimum order that belongs to $\mathcal{R}(2K_2, F_n)$, for $n \geq 3$. We have also found all members of $\mathcal{R}(2K_2, F_n)$ with a minimum order for $n \in [3, 7]$.

Key words and Phrases: dominating vertex, fan graph, matching, ramsey minimal graph.

1. INTRODUCTION

Frank Plumpton Ramsey formulated Ramsey's theory to assist in determining the truth or falsehood of logical formulas [1]. Ramsey's theory evolved into minimal Ramsey graph theory, as defined by Burr et al. in 1976 [2]. This paper regards all graphs as simple and utilizes the definition as referenced in [3]. The notation $F \rightarrow (G, H)$ signifies that any red-blue edge coloring of the graph F leads to F containing either a red subgraph isomorphic to G or a blue subgraph isomorphic to H.

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In graph theory, a graph F serves as a Ramsey graph for a pair (G, H) if it satisfies the condition $F \rightarrow (G, H)$. When F is the complete graph K_n, the problem of identifying the smallest integer n such that $K_n \rightarrow (G, H)$ has been thoroughly researched [4, 5, 6, 7]. This smallest n is referred to as the (graph) Ramsey number for the pair (G, H), denoted by $r(G, H)$. A graph F is referred to as a *Ramsey graph* (G, H)*-minimal* if it holds that $F \rightarrow (G, H)$, but for any subgraph formed by removing an edge, $F - e \nrightarrow (G, H)$. All such (G, H)-minimal graphs are grouped into a class called the *Ramsey class* (G, H)*-minimal*, represented as $\mathcal{R}(G, H)$.

The pair (G, H) is referred to as Ramsey-finite if the set $\mathcal{R}(G, H)$ is finite; otherwise, it is termed Ramsey-infinite. Łuczak [8] demonstrated that the set $\mathcal{R}(G, H)$ is infinite for any forest G other than a matching and for any graph H that contains a cycle. Furthermore, Burr et al. [9] established that $\mathcal{R}(mK_2, H)$ is Ramsey finite for every graph H and any positive integer m. Borowiecki et al. [5] provided a characterization of the graphs in $R(K_{1,2}, K_{1,m})$ where $m \geq 3$. Numerous papers have explored the characterization of infinite families of Ramsey $(K_{1,2}, C_4)$-minimal graphs (refer to [10, 11, 12]). Yulianti et al. [13] constructed some infinite classes of Ramsey $(K_{1,2}, P_4)$-minimal graphs. Subsequently, Borowiecki et al. [14] identified the graphs in $R(K_{1,2}, K_3)$. Borowiecka-Olszewska and Haluszczak [15] introduced a method to generate an infinite family of Ramsey $(K_{1,m}, G)$-minimal graphs, where $m \geq 2$ and G belongs to a family of 2-connected graphs.

In this paper, we focus on Ramsey-finite. Mangersen and Oeckermann [16] proved that $\mathcal{R}(2K_2, K_{1,2}) = \{2K_{1,2}, C_4, C_5\}$, and presented the characterization of graphs belonging to $\mathcal{R}(2K_2, K_{1,n})$, for $n \geq 3$. Furthermore, Muhshi and Baskoro [17] proved that $\mathcal{R}(3K_2, P_3) = \{3P_3, C_4 \cup P_3, C_5 \cup P_3, C_7, C_8\}$. Baskoro and Yulianti [18] characterized all graphs in $\mathcal{R}(2K_2, P_n)$ for $n = 4, 5$, Tatanto and Baskoro [19] characterized graphs in $\mathcal{R}(2K_2, 2P_n)$, and Silaban et al. [20] characterized graphs in $\mathcal{R}(mK_2, P_4)$. [21] characterized all graphs belonging to $\mathcal{R}(2K_2, K_4)$, and in [22], Wijaya et al. characterized all unicyclic graphs belonging to $\mathcal{R}(mK_2, P_3)$.

Baskoro and Wijaya [23] derived the necessary and sufficient conditions for the graphs to be in $\mathcal{R}(2K_2, H)$ for any connected graph H. They proved the following theorem.

Theorem 1.1. [23] *Let H be any connected graph. $F \in \mathcal{R}(2K_2, H)$ if and only if the following conditions are satisfied:*

(i) for every $v \in V(F)$, $F - v \supseteq H$,
(ii) for every K_3 in F, $F - E(K_3) \supseteq H$,
(iii) for every $e \in E(F)$, there exists $v \in V(F)$ or K_3 in F such that $(F - e) - v \not\supseteq H$ or $(F - e) - E(K_3) \not\supseteq H$.

If a graph F conforms to both Theorem 1.1(i) and (ii), it follows that $F \rightarrow (2K_2, H)$. Moreover, should a graph F fulfill Theorem 1.1(iii), it implies that F possesses the minimality property of F, meaning for every edge e in F, $F - e \nrightarrow (2K_2, H)$. In addition, Wijaya and Baskoro [24] identified the necessary and sufficient conditions for graphs within $\mathcal{R}(mK_2, H)$. In addition, Rafif et al. [25] discovered several results concerning Ramsey $(2K_2, G)$-minimal graphs of the smallest order, where the graph

G possesses a dominating vertex (a vertex adjacent to all other vertices) and has an independence number of at least 2, and they also proved the complete list of $\hat{\mathcal{R}}(2K_2, W_n)$ for $n \in [3, 7]$, where $W_n = K_1 + C_n$. In the same year, Rafif et al. [26] also proved the complete list of $\hat{\mathcal{R}}(2K_2, \mathrm{Fr}_n)$ for $n \geq 2$, where $\mathrm{Fr}_n = K_1 + nK_2$.

A fan graph F_n is defined as the graph $P_n + K_1$, where P_n is the path on n vertices. In this paper, we propose some properties that are sufficient or/and necessary conditions for membership of $\mathcal{R}(2K_2, F_n)$ with minimum order, for $n \geq 3$. We also determine all non-isomorphic Ramsey $(2K_2, F_n)$-minimal graphs, for $n \in [3, 7]$.

2. MAIN RESULTS

We will introduce some definitions and notations. A union of m disjoint copies of K_2, denoted by mK_2, is called a *matching.* A complete multipartite graph composed of j partite sets with k vertices within each set is designated by $K_{j\times k}$. The notation $[i, t]$ represents the set of all integers that lie between i and t (including i and t themselves). Given a graph G, E_i denotes the set of i arbitrary edges of $E(G)$. The notation $G - E_i$ denotes the removal of any i edges from the graph G. The degree of a vertex v in a graph G is denoted by $\deg_G(v)$. The maximum and minimum degrees of the graph G are represented by $\Delta(G)$ and $\delta(G)$, respectively. The complement of a graph G, denoted by $\bar{G}$, is a graph that has the same vertices as G, but $uv \in E(G)$ if and only if $uv \notin E(\bar{G})$.

2.1. Some Properties of $\mathcal{R}(2K_2, F_n)$ with Minimum Order.

In this subsection, we discuss some properties of the graph F that satisfy $F \in \hat{\mathcal{R}}(2K_2, F_n)$ for $n \geq 3$. From Theorem 1.1, we obtain some properties of graph $F \in R(aK_2, F_n)$ in the following Lemma 2.1.

Lemma 2.1. *If $F \in \mathcal{R}(2K_2, F_n)$, where $n \geq 3$, then following conditions are satisfied:*

(i). $|V(F)| \geq n + 2$

(ii). $\Delta(F) \geq n$

Proof. Let $v \in V(F)$.

(i). Suppose $|V(F)| < n + 2$. Since $|V(F)| < n + 2$, then $|V(F - v)| < n + 1 = |V(F_n)|$, thus, $F - v \not\supseteq F_n$, for $n \geq 3$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, $V(F) \geq n + 2$.

(ii). Suppose $\Delta(F) < n$. Since $\Delta(F) < n$, then $\Delta(F - v) < n = \Delta(F_n)$. Thus, $F - v \not\supseteq F_n$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, $\Delta(F) \geq n$.

□

From Lemma 2.1, we know that the maximum degree of a member of $\mathcal{R}(2K_2, F_n)$ with minimum order is either n or $n+1$. In the following Proposition 2.2, we obtain a characterization of graphs that are members of $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree n, for odd $n \geq 3$.

Proposition 2.2. *For odd $n \geq 3$, there are no graphs belong to $\mathcal{R}(2K_2, F_n)$ with minimum order $n+2$ and maximum degree n.*

Proof. Let $v_i \in E(K_{n+2})$, for $i \in [1, n+2]$. Let H be any graph with order $n+2$ and maximum degree n. For odd $n \geq 3$, define a graph $F = K_{n+2} - E \supseteq H$, where $E = \{v_{2j-1}v_{2j} | j \in [1, \frac{n+1}{2}]\} \cup \{v_1 v_{n+2}\}$. Consider the graph $F - v_{n+2}$, $\Delta(F - v_{n+2}) = n - 1$. Thus, $F - v_{n+2} \not\supseteq F_n$. By Theorem 1.1, $F \not\rightarrow (2K_2, F_n)$, for odd $n \geq 3$. Since $F \supseteq H$, then $H \not\rightarrow (2K_2, F_n)$. Therefore, $H \notin \mathcal{R}(2K_2, F_n)$. $\square$

In the following Theorem 2.3, we obtain a characterization of graphs that are members of $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree n, for even $n \geq 4$.

Theorem 2.3. *For even $n \geq 4$, $K_{(\frac{n}{2}+1)\times 2}$ is the unique graph in $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree n.*

Proof. Let $v_i \in V(K_{n+2})$ for $i \in [1, n+2]$. For even $n \geq 4$, define the graph $K_{(\frac{n}{2}+1)\times 2} = K_{n+2} - E$, where $E = \{v_{2j-1}v_{2j} \mid j \in [1, n+1]\}$. First, we will show that $K_{(\frac{n}{2}+1)\times 2} \in \mathcal{R}(2K_2, F_n)$.

(1) We will show that $K_{(\frac{n}{2}+1)\times 2} \rightarrow (2K_2, F_n)$. Assume that there is no red $2K_2$, then the subgraph induced by the red edges is either a $K_{1,n}$ or a C_3.
   (i) Red subgraph $K_{1,n}$.
   Without loss of generality, color every edge incident to vertex v_1 with red. Next, color every edge incident to vertex v_2 with blue, making v_2 the dominating vertex in the graph F_n. Consider the uncolored edges in the graph $K_{(\frac{n}{2}+1)\times 2}$ are $K_{(\frac{n}{2}+1)\times 2} - E(K_{1,n}) - v_2 = K_{\frac{n}{2}\times 2}$ (see Figure 1 (i)). Since $K_{\frac{n}{2}\times 2}$ contains a P_n, there exists a blue subgraph F_n in $K_{\frac{n}{2}+1\times 2}$.
   (ii) Red subgraph C_3.
   Without loss of generality, color every edge of the triangle formed by vertices v_1, v_3, and v_5 with red. Next, color every edge incident to vertex v_2 with blue, making v_2 the dominating vertex in the graph F_n. Since v_1 and v_2 are not adjacent, vertex v_1 can be disregarded. Consider the uncolored edges in the graph $K_{(\frac{n}{2}+1)\times 2}$ are $K_{(\frac{n}{2}+1)\times 2} - E(C_3) - v_2 - v_1 = K_{\frac{n}{2}\times 2} - e$ (see Figure 1 (ii)). Since $K_{\frac{n}{2}\times 2} - e$ contains a P_n, there exists a blue subgraph F_n in $K_{\frac{n}{2}+1\times 2}$.

   In both cases above, we always obtain a blue subgraph F_n in $K_{\frac{n}{2}+1\times 2}$. Therefore, $K_{(\frac{n}{2}+1)\times 2} \rightarrow (2K_2, F_n)$.

(2) Next, we will show that $K_{(\frac{n}{2}+1)\times 2} - e \not\rightarrow (2K_2, F_n)$. Without loss of generality, let $e = v_1 v_3$. Color every edge incident to vertex v_2 with red. Consider the uncolored edges in the graph $K_{(\frac{n}{2}+1)\times 2} - e$ have a maximum degree $n - 1$ (see Figure 1 (iii)). Thus, there does not exist a blue subgraph F_n in $K_{(\frac{n}{2}+1)\times 2} - e$. Therefore, $K_{(\frac{n}{2}+1)\times 2} - e \not\rightarrow (2K_2, F_n)$.

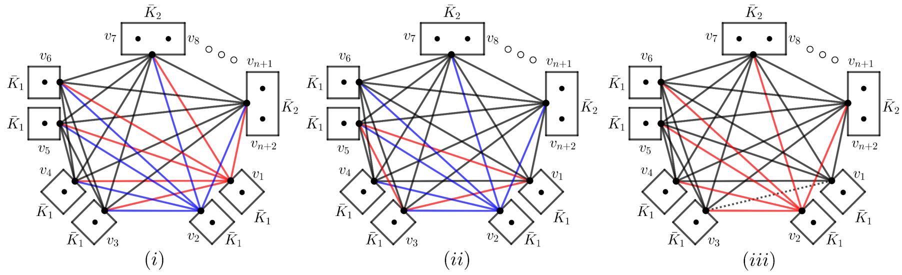

FIGURE 1. Red-Blue Coloring of $K_{(\frac{n}{2}+1)\times 2}$ and $K_{(\frac{n}{2}+1)\times 2} - e$

From (1) and (2), we have proven that $K_{(\frac{n}{2}+1)\times 2} \in \mathcal{R}(2K_2, F_n)$.

Next, we will show that $K_{(\frac{n}{2}+1)\times 2}$ is the only graph in $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree n. Suppose $F \neq K_{(\frac{n}{2}+1)\times 2}$. Since $\Delta(F) = n$, $|V(F)| = n + 2$, and $F \neq K_{(\frac{n}{2}+1)\times 2}$, then $F \subset K_{(\frac{n}{2}+1)\times 2}$. Since $K_{(\frac{n}{2}+1)\times 2} \in \mathcal{R}(2K_2, F_n)$ and $F \subset K_{(\frac{n}{2}+1)\times 2}$, then $F \notin \mathcal{R}(2K_2, F_n)$. Hence, we have proven that $K_{(\frac{n}{2}+1)\times 2}$ is the only graph in $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree n. □

In the following Lemma 2.4, we obtain a property of graphs that are members of $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree $n + 1$.

Lemma 2.4. *Let $F \in \mathcal{R}(2K_2, F_n)$, where $n \geq 3$. If $|V(F)| = n + 2$, and $\Delta(F) = n+1$ then there are at least four vertices with degrees of at least n, and at least two of them are of degrees $n + 1$.*

Proof. Suppose there is at most one vertex with degree $n + 1$. Let $v \in V(F)$, $\deg(v) = \Delta(F) = n + 1$. Since $\Delta(F - v) \leq n - 1 < n = \Delta(F_n)$, thus $F_n \nsubseteq F - v$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, there are at least two vertices with degree $n+1$. Next, suppose there are at most three vertices with a degree of at least n, we know that at least two of them have a degree of $n + 1$. Let $v_1, v_2, v_3 \in V(F)$, with $\deg(v_1) = \deg(v_2) = n + 1$ and $\deg(v_3) = n$. Observe a K_3, with $V(K_3) = \{v_1, v_2, v_3\}$. Consider $\Delta(F - E(K_3)) = n - 1 < n = \Delta(F_n)$, thus $F_n \nsubseteq F - E(K_3)$. By Theorem 1.1 (ii), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, there are at least four vertices with a degree of at least n. □

From Lemma 2.1, we obtain some necessary conditions for graphs that are members of $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree $n + 1$ in the following Theorem 2.5.

Theorem 2.5. *Let $F \in \mathcal{R}(2K_2, F_n)$, with $n \geq 5$. If $|V(F)| = n + 2$ and $\Delta(F) = n + 1$, then all the following conditions are necessary for the graph F.*

(1) The minimum degree of $F, \delta(F) \geq 3$.
(2) The maximum number of vertices of degree three is two.

(3) If there are exactly two vertices of degree $n+1$, exactly two vertices of degree n, and there is a vertex of degree four, then the vertices of degree n are adjacent.

(4) For $n \geq 6$, if there are exactly two vertices of degree $n+1$ and exactly two vertices of degree n, then there are at most two vertices of maximum degree four.

(5) $F \not\supseteq \bar{K}_{\frac{n}{2}+1}$.

Proof. Given that $F \in \mathcal{R}(2K_2, F_n)$, with $n \geq 5$ and $|V(F)| = n+2$. Without loss of generality, let $v_1, v_2, v_3, v_4 \in V(F)$. By Lemma 2.1, $\Delta(F) = \deg(v_1) = \deg(v_2) = n+1$ and $\deg(v_3) = \deg(v_4) \geq n$.

(1) Let $v_5 \in V(F)$. Suppose $\deg(v_5) = \delta(F) = 2$. Consider $\delta(F - v_1) = \deg(v_5) - 1 = 1$ and $\delta(F_n) = 2$, hence $v_5 \notin V(F_n)$. Since $|V(F - v_1)| = n+1$ and $v_5 \notin V(F_n)$, then $F_n \not\subseteq F - v_1$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, $\deg(v_5) = \delta(F) \geq 3$.

(2) Let $v_i \in V(F)$, for $i \in [5, 7]$ where $\deg(v_5) = \deg(v_6) = 3$. Suppose $\deg(v_6) = 3$ for $i \in [5, 7]$. Consider $\delta(F - v_1) = \deg(v_i) - 1 = 2$ and $\Delta(F) = \deg(v_2) = n$. Without loss of generality, v_2 is a dominating vertex in graph F_n. Next, observe that $\delta((F - v_1) - v_2) = \deg(v_i) - 2 = 1$. Since $|V((F - v_1) - v_2)| = n$ and $\deg(v_i) = 1$ in $(F - v_1) - v_2$, thus $P_n \not\subseteq (F - v_1) - v_2$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, $\deg(v_7) \geq 4$.

(3) Let $\deg(v_3) = \deg(v_4) = n$. Let $v_5 \in V(F)$ with $\deg(v_5) = 4$. Suppose v_3 and v_4 are adjacent. Consider a complete graph K_3 with $V(K_3) = [v_1, v_2, v_5]$, $\delta(F - E(K_3)) = \deg(v_5) = 2$, and $\Delta(F) = \deg(v_3) = n$. Without loss of generality, v_3 is a dominating vertex in graph F_n. Next, observe that $\delta((F - E(K_3)) - v_3) = \deg(v_5) = 1$. Since v_3 and v_4 are not adjacent and v_3 is a dominating vertex of the wheel graph F_n, thus $v_4 \notin F_n$, hence $\deg(v_5) = 0$ in $(F - E(K_3)) - v_3 - v_4$. Since $\deg(v_5) = 0$ in $(F - E(K_3)) - v_3 - v_4$ and $|V(F - E(K_3)) - v_3| = n$, thus $P_n \not\subseteq V(F - E(K_3)) - v_3$. By Theorem 1.1 (ii), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, v_3 and v_4 are adjacent.

(4) Let $\deg(v_3) = \deg(v_4) = n$. Let $v_i \in V(F)$ for $i \in [5, 6, 7]$ where $\deg(v_5) = \deg(v_6) = 4$. Suppose $\deg(v_5) = 4$. Consider $\delta(F - v_1) = \deg(v_i) - 1 = 3$ for $i \in [5, 6, 7]$ and $\Delta(F) = \deg(v_2) = n$, then v_2 is a dominating vertex in graph F_n. Next, observe that $\delta((F - v_1) - v_2) = \deg(v_i) - 2 = 1$. Since each vertex v_i is adjacent only to v_3 and v_4 and $|V((F - v_1) - v_2)| = n$, thus $P_n \not\subseteq V((F - v_1) - v_2)$ for $n \geq 6$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, $\deg(v_5) \geq 5$

(5) Suppose $F \subseteq \bar{K}_{\lceil \frac{n}{2} \rceil + 1}$. Consider $\Delta(F - v_1) = \deg(v_2) = n$, thus v_2 is the dominating vertex of the wheel graph F_n. Since $F \subseteq \bar{K}_{\lceil \frac{n}{2} \rceil + 1}$, there exists a partition with $\lceil \frac{n}{2} + 1 \rceil$ vertices in the graph F. Since $|V(F - v_1 - v_2)| = n$ and there is a partition with $\lceil \frac{n}{2} + 1 \rceil$ vertices in $F - v_1 - v_2$, then $\lceil \frac{n}{2} \rceil + 1 > n - \lceil \frac{n}{2} \rceil - 1$. Consequently, $P_n \not\subseteq F - v_1 - v_2$. By Theorem 1.1 (i), $F \notin \mathcal{R}(2K_2, F_n)$, a contradiction. Therefore, $F \not\supseteq \bar{K}_{\frac{n}{2}+1}$.

□

To facilitate the application of the sufficient condition in Theorem 2.5 above, we can define several graph sets in Remark 2.6 as follows.

Remark 2.6. *Define* ${\mathcal{G}_{\mathcal{N}}}^i$ *as a collection of all graphs of order* $n+2$ *satisfying Theorem 2.5* (i), *for* $i \in [1,5]$.

2.2. The Complete List of $\mathcal{R}(2K_2, F_n)$ with Minimum Order, for $n \in [3,7]$.

In this subsection, we give the complete list of $\hat{\mathcal{R}}(2K_2, F_n)$ for $n \in [3,7]$ in Theorems 2.7-2.11. We characterize of the member $\mathcal{R}(2K_2, F_3)$ with minimum order in the following Theorem 2.7.

Theorem 2.7. *The graph* K_5 *is the unique graph with five vertices in* $\mathcal{R}(2K_2, F_3)$.

Proof. We will show that $K_5 \to (2K_2, F_3)$. Suppose that there is no red $2K_2$, then the subgraph induced by red edges is either $K_{1,4}$ or C_3. For both possibilities, a blue F_3 always appears in the coloring (see Figure 2 $K_5 \to (2K_2, F_3)$). Thus, $K_5 \to (2K_2, F_3)$. Next, we will show that $K_5 - e \nrightarrow (2K_2, F_3)$. Let the red edges

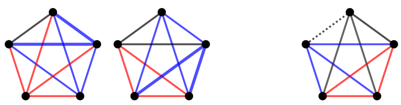

FIGURE 2. Red-Blue Coloring of Graphs K_5 and $K_5 - e$

induce a K_3 with vertices of degree 4. Consider that the subgraph with blue edges is $K_{3,2}$ (see Figure 2 $K_5 - e \not\to (2K_2, F_3)$). Since $K_{3,2} \not\supset F_3$, this blue subgraph does not contain F_3. Thus, $K_5 - e \nrightarrow (2K_2, F_3)$. It is proven that $K_5 \in \mathcal{R}(2K_2, F_3)$. Furthermore, since $K_5 - E_i \subset K_5$, for $i \geq 1$, then $K_5 - E_i \notin \mathcal{R}(2K_2, F_3)$. Therefore, K_5 is the unique graph with five vertices in $\mathcal{R}(2K_2, F_3)$.

□

Henceforth, we will only provide a red-blue coloring figure on graph F to demonstrate that $F \in \mathcal{R}(2K_2, W_n)$, for $n \geq 4$. In the following Theorem 2.8, we obtain the characterization of the member $\mathcal{R}(2K_2, F_4)$ with minimum order.

Theorem 2.8. *The graph* $K_{3\times 2}$ *is the unique graph with six vertices in* $\mathcal{R}(2K_2, F_4)$.

Proof. By Theorem 2.3, $K_{3\times 2}$ is the unique graph in $\mathcal{R}(2K_2, F_4)$ with maximum degree 4. Thus, we only need to show that there is no graph with maximum degree 5 as a member of $\mathcal{R}(2K_2, F_4)$. Let $\mathcal{F}_R^y = \{K_6 - E_y \mid K_6 - E_y \in \mathcal{R}(2K_2, F_4)\}$. Since there is no graph $K_6 - E_y$ for $y > 3$ that satisfies Lemma 2.4, then $y \leq 3$.

Case 1. $K_6 - E_3$.
The graphs that satisfy Lemma 2.4 are only $K_6 - E(P_4)$ and $K_6 - E(K_{1,3})$. Since $K_6 - E(P_4) \notin \mathcal{G_N}^3$ and $K_6 - E(K_{1,3}) \notin \mathcal{G_N}^1$, then $\mathcal{F}_R^3 = \emptyset$. Therefore, there are no graphs obtained by removing any three edges from K_6 that belong to $\mathcal{R}(2K_2, F_4)$.

Case 2. $K_6 - E_2$.
The graphs that satisfy Lemma 2.4 are only $K_6 - E(P_3)$ and $K_6 - E(2K_2)$. By Figure 3, $K_6 - E(P_3) \not\to (2K_2, F_4)$ and since $K_{3\times 2} \supset K_6 - E(2K_2)$, then $\mathcal{F}_R^2 = \emptyset$. Therefore, there are no graphs obtained by removing any two edges from K_6 that belong to $\mathcal{R}(2K_2, F_4)$.

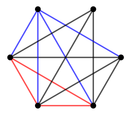

FIGURE 3. $K_6 - E(P_3) \not\to (2K_2, F_4)$

Case 3. $K_6 - E_k$, for $k \leq 1$.
Since $K_{3\times 2} \supset K_6 - e$, then $\mathcal{F}_R^k = \emptyset$, for $k \leq 1$. Therefore, there are no graphs obtained by removing at most one edge from K_6 that belong to $\mathcal{R}(2K_2, F_4)$.

Based on Case $1-3$, there is no graph in $\mathcal{R}(2K_2, F_4)$ with six vertices and maximum degree five. $\square$

The graphs $F_5^{\mathcal{R}}$ is shown in Figure 4.

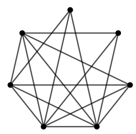

FIGURE 4. Graph $F_5^{\mathcal{R}}$

In the following Theorem 2.9, we obtain the characterization of the member $\mathcal{R}(2K_2, F_5)$ with minimum order.

Theorem 2.9. *The graph $F_5^{\mathcal{R}}$ is the unique graph with seven vertices in $\mathcal{R}(2K_2, F_5)$.*

Proof. Consider the red-blue coloring on $F_5^{\mathcal{R}}$ and $F_5^{\mathcal{R}} - e$ as shown in Figure 5. Thus, $F_5^{\mathcal{R}} \in \mathcal{R}(2K_2, F_5)$. By Proposition 2.2 there is no graph with maximum degree 5 in $\mathcal{R}(2K_2, F_5)$. Thus, we only need to show that $F_5^{\mathcal{R}}$ is the only graph with maximum degree 6 in $\mathcal{R}(2K_2, F_5)$. Let $\mathcal{F}_R^y = \{K_7 - E_y | K_7 - E_y \in \mathcal{R}(2K_2, F_5)\}$. Since there is no graph $K_7 - E_y$ for $y > 5$ that satisfies Lemma 2.4, then $y \leq 5$.

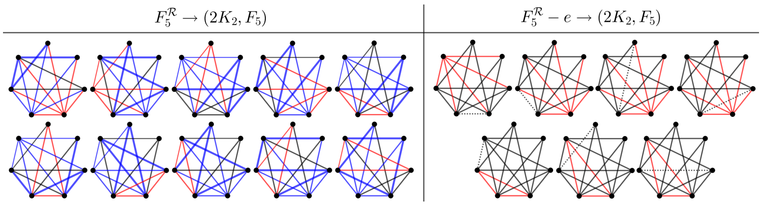

FIGURE 5. Red-Blue Coloring of Graphs $F_5^{\mathcal{R}}$ and $F_5^{\mathcal{R}} - e$

Case 1. $K_7 - E_5$.

A graph that satisfies Lemma 2.4 other than $F_5^{\mathcal{R}}$ is only $K_7 - E_5^1$, for a particular set of five edges E_5^1 (see Figure 6). Since $K_7 - E_5^1 \notin {\mathcal{G}_{\mathcal{N}}}^1$, by Theorem 2.5, $K_7 - E_5^1 \notin \mathcal{R}(2K_2, F_5)$. Thus, $\mathcal{F}_R^5 = \{F_5^{\mathcal{R}}\}$. Therefore, removing any five edges from the graph K_7 only produces the graphs $F_5^{\mathcal{R}}$ as members of $\mathcal{R}(2K_2, F_7)$.

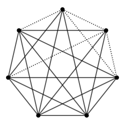

FIGURE 6. Graphs $K_7 - E_5^1$

Case 2. $K_7 - E_4$.

Figure 7 shows all possible graphs that satisfy Lemma 2.4, for particular sets of four edges E_4^i, for $i \in [1, 5]$. Since $K_7 - E_4^1 \notin {\mathcal{G}_{\mathcal{N}}}^3$ and $K_7 - E_4^2 \notin {\mathcal{G}_{\mathcal{N}}}^1$ by Theorem 2.5, $K_7 - E_4^1, K_7 - E_4^2 \notin \mathcal{R}(2K_2, F_5)$. Since $K_7 - E_4^j \supset F_5^{\mathcal{R}}$, for $j \in [3, 5]$, then $K_7 - E_4^j \notin \mathcal{R}(2K_2, F_5)$. Thus, $\mathcal{F}_R^4 = \emptyset$. Therefore, there are no graphs obtained by removing any four edges from K_7 that belong to $\mathcal{R}(2K_2, F_4)$.

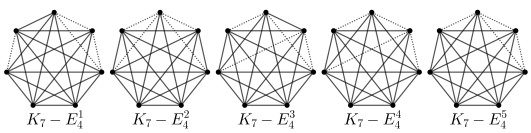

FIGURE 7. Graphs $K_7 - E_4^i$, for $i \in [1, 5]$

Case 3. $K_7 - E_3$.

In this case we explore the graphs $K_7 - E_4^j + e$, for $j = [1, 2]$ and any edge e. We obtain three non-isomorphic graphs $K_7 - E_3^i$, for $i \in [1, 3]$, as depicted in Figure 8. Since $K_7 - E_3^i \supset F_5^{\mathcal{R}}$, for $i \in [1, 3]$, then $K_7 - E_3^i \notin \mathcal{R}(2K_2, F_5)$. Thus, $\mathcal{F}_R^3 = \emptyset$. Therefore, there are no graphs obtained by removing any three edges from K_7 that belong to $\mathcal{R}(2K_2, F_5)$.

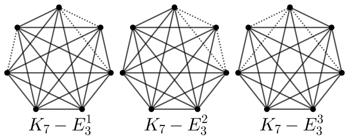

Figure 8. Graphs $K_7 - E_3^i$, for $i \in [1, 3]$

Case 4. $K_7 - E_t$, for $t \leq 2$.
Since $K_7 - E_3^s + E_k \supset F_5^{\mathcal{R}}$, for $k \leq 3$, then $K_7 - E_k \notin \mathcal{R}(2K_2, F_5)$. Thus, $\mathcal{F}_R^k = \emptyset$, for $k \leq 2$. Therefore, there are no graphs obtained by removing at most any two edges from K_6 that belong to $\mathcal{R}(2K_2, F_5)$.

Based on Case $1-4$, $F_5^{\mathcal{R}}$ is the unique graph with six vertices and maximum degree 6 in $\mathcal{R}(2K_2, F_5)$. □

The graphs $F_{6,i}^{\mathcal{R}}$ for $i \in [1, 5]$ are shown in Figure 9.

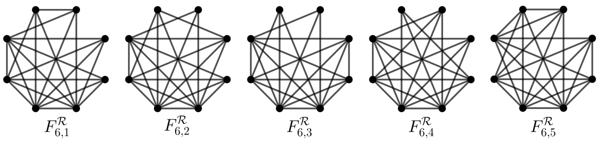

Figure 9. $F_{6,i}^{\mathcal{R}}$ for $i \in [1, 5]$

In the following Theorem 2.10, we obtain the characterization of the member $\mathcal{R}(2K_2, F_6)$ with minimum order.

Theorem 2.10. *The graphs with eight vertices in $\mathcal{R}(2K_2, F_6)$ are precisely $K_{4\times 2}$ and $F_{6,i}^{\mathcal{R}}$ for $i \in [1, 5]$.*

Proof. By Theorem 2.3, $K_{4\times 2}$ is the unique graph in $\mathcal{R}(2K_2, F_6)$ with maximum degree 4. Next, the red-blue colorings of $F_{6,i}^{\mathcal{R}}$ and $F_{6,i}^{\mathcal{R}} - e$ for $i \in [1, 5]$ as shown in Figure 10. Therefore, we obtain $K_{4\times 2}$, $F_{6,i}^{\mathcal{R}} \in \mathcal{R}(2K_2, F_6)$. Thus, we only need to show that $F_{6,i}^{\mathcal{R}}$ for $i \in [1, 5]$ is the only graphs with maximum degree 8 in $\mathcal{R}(2K_2, F_6)$. Let $\mathcal{F}_R^y = \{K_9 - E_y | K_9 - E_y \in \mathcal{R}(2K_2, F_6)\}$. Since there is no graph $K_8 - E_y$ for $y > 7$ that satisfies Lemma 2.4 and Theorem 2.5, then $y \leq 7$.

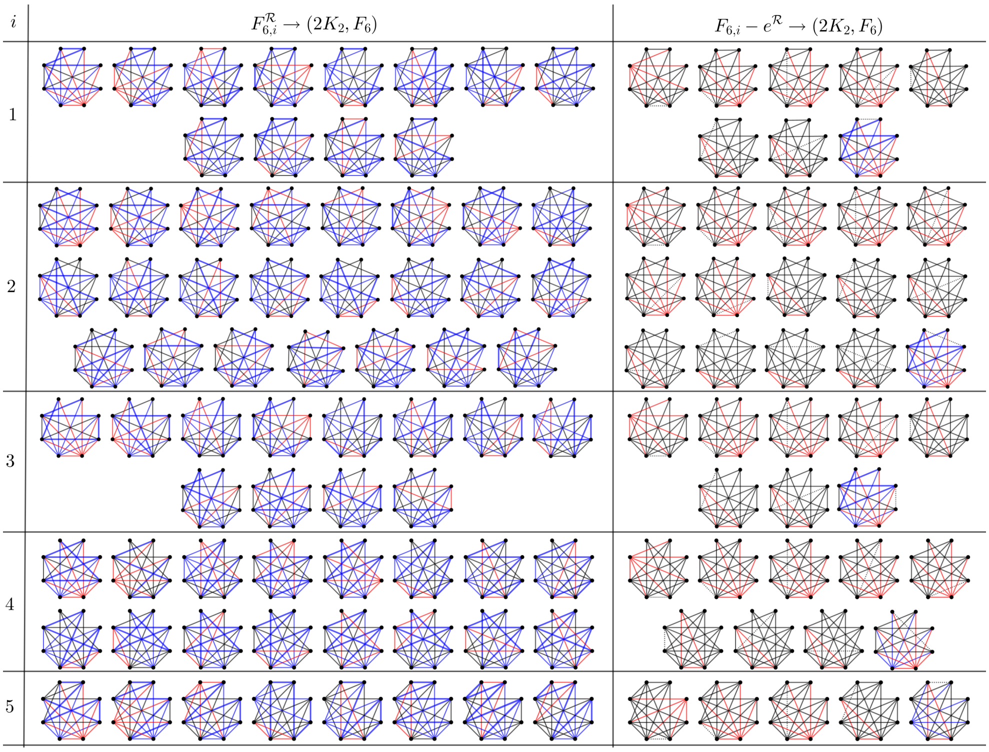

FIGURE 10. Red-Blue Coloring of Graphs $F^{\mathcal{R}}_{6,i}$ and $F^{\mathcal{R}}_{6,i} - e$, for $i \in [1, 5]$

Case 1. $K_8 - E_7$

The graphs that satisfies Lemma 2.4 other than $F^{\mathcal{R}}_{6,i}$, $i \in [1, 4]$, are only $K_8 - E^j_7$, for particular sets of seven edges E^j_7, for $j \in [1, 3]$ (see Figure 11). Since $K_8 - E^1_7, K_8 - E^2_7 \notin \mathcal{G_N}^4$ and $K_8 - E^3_7 \notin \mathcal{G_N}^1$, by Theorem 2.5, $K_8 - E^j_7 \notin \mathcal{R}(2K_2, F_6)$. Thus $\mathcal{F}^7_R = \{F^{\mathcal{R}}_{6,i} | i \in [1, 4]\}$. Therefore, removing any seven edges from the graph K_8 only produces the graphs $F^{\mathcal{R}}_{6,i}$ for $i \in [1, 4]$ as members of $\mathcal{R}(2K_2, F_7)$.

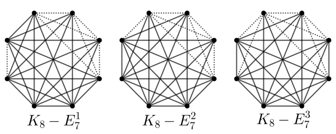

FIGURE 11. Graphs $K_8 - E^i_7$, for $i \in [1, 3]$

Case 2. $K_8 - E_6$

In this case we explore the graphs $K_8 - E_7^j + e$ for $j \in [1,3]$ and any edge e. We obtain seven non-isomorphic graphs $K_8 - E_6^i$, for $i \in [1,7]$, as depicted in Figure 12. Since $K_8 - E_6^1 \notin \mathcal{G_N}^4$, $K_8 - E_6^2 \notin \mathcal{G_N}^3$, and $K_8 - E_7^3 \notin \mathcal{G_N}^1$, by Theorem 2.5, $K_8 - E_6^{t_1} \notin \mathcal{R}(2K_2, F_6)$, for $t_1 \in [1,3]$. Since $K_8 - E_6^4 \supset F_{6,3}^{\mathcal{R}}$ and $K_8 - E_6^{t_2} \supset F_{6,4}^{\mathcal{R}}$, for $t_2 \in [5,7]$, then $K_8 - E_6^{t_2} \notin \mathcal{R}(2K_2, F_6)$ for $t_2 \in [4,7]$. Thus, $\mathcal{F}_R^6 = \emptyset$. Therefore, there are no graphs obtained by removing any six edges from K_8 that belong to $\mathcal{R}(2K_2, F_6)$.

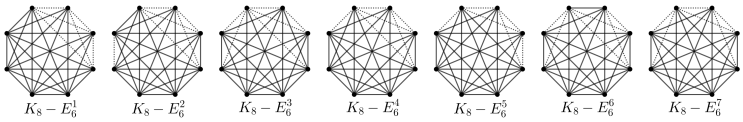

FIGURE 12. Graphs $K_8 - E_6^i$, for $i \in [1,7]$

Case 3. $K_8 - E_5$

In this case we explore the graphs $K_8 - E_6^j + e$ for $j \in [1,3]$ and any edge e. We obtain six non-isomorphic graphs $F_{6,5}^{\mathcal{R}}$ and $K_8 - E_5^i$, for $i \in [1,5]$, as depicted in Figure 9 and 13. Since $K_8 - E_5^1 \notin \mathcal{G_N}^1$, by Theorem 2.5, $K_8 - E_5^1 \notin \mathcal{R}(2K_2, F_6)$. Since $K_8 - E_5^2 \supset F_{6,1}^{\mathcal{R}}$, $K_8 - E_5^3 \supset F_{6,2}^{\mathcal{R}}$, and $K_8 - E_5^s \supset F_{6,4}^{\mathcal{R}}$ for $s \in [4,5]$, then $K_8 - E_5^t \notin \mathcal{R}(2K_2, F_6)$ for $t \in [1,5]$. Thus, $\mathcal{F}_R^5 = \{F_{6,5}^{\mathcal{R}}\}$. Therefore, removing any five edges from the graph K_7 only produces the graph $F_{6,5}^{\mathcal{R}}$ as member of $\mathcal{R}(2K_2, F_6)$.

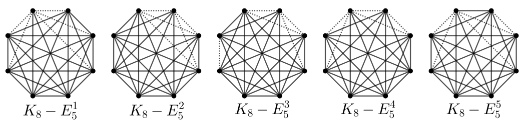

FIGURE 13. Graphs $K_8 - E_5^i$, for $i \in [1,5]$

Case 4. $K_8 - E_k$, for $t \leq 4$

Since $K_8 - E_5^1 + E_i \supset F_{6,4}^{\mathcal{R}}$, for $i \leq 5$ then $K_8 - E_5^1 + E_i \notin \mathcal{R}(2K_2, F_6)$. Thus, $\mathcal{F}_R^t = \emptyset$, for $t \leq 4$. Therefore, there are no graphs obtained from removing at most four edges from K_8 as a member of $\mathcal{R}(2K_2, F_6)$.

Based on Case 1 – 4, the graphs with eight vertices and maximum degree 7 in $\mathcal{R}(2K_2, F_6)$ are only $F_{6,i}^{\mathcal{R}}$, for $i \in [1,5]$. $\square$

The graphs $F^{\mathcal{R}}_{7,i}$ for $i \in [1, 12]$ are shown in Figure 14.

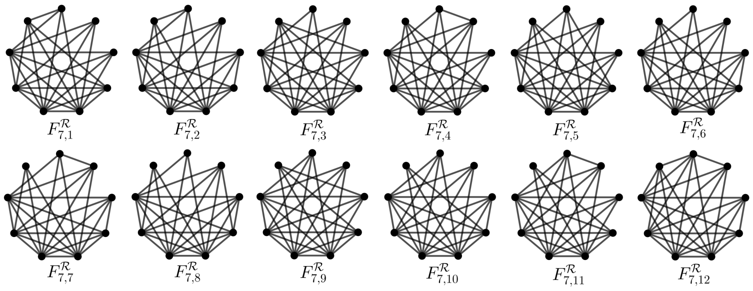

FIGURE 14. $F^{\mathcal{R}}_{7,i}$ for $i \in [1, 12]$

In the following Theorem 2.11, we obtain characterization of the member $\mathcal{R}(2K_2, F_7)$ with minimum order.

Theorem 2.11. *The graphs with nine vertices in $\mathcal{R}(2K_2, F_7)$ are precisely $F^{\mathcal{R}}_{7,i}$ for $i \in [1, 12]$.*

Proof. The red-blue colorings of $F^{\mathcal{R}}_{7,i}$ and $F^{\mathcal{R}}_{7,i} - e$ for $i \in [1, 12]$ as shown in Figure 16. Thus, $F^{\mathcal{R}}_{7,i} \in \mathcal{R}(2K_2, F_7)$.

By Proposition 2.2 there is no graph with maximum degree 7 in $\mathcal{R}(2K_2, F_7)$. Therefore, we only need to show that $F^{\mathcal{R}}_{7,i}$ for $i \in [1, 12]$ is the only graphs with maximum degree 8 in $\mathcal{R}(2K_2, F_5)$. Let $\mathcal{F}^{y}_{R} = \{K_9 - E_y | K_9 - E_y \in \mathcal{R}(2K_2, F_7)\}$. Since there is no graph $K_9 - E_y$ for $y > 11$ that satisfies Lemma 2.4 and Theorem 2.5, then $y \leq 11$

Case 1. $K_9 - E_{11}$

The graphs that satisfies Lemma 2.4 are only $K_9 - E^{i}_{11}$, for particular sets of 11 edges E^{i}_{11}, for $i \in [1, 7]$ (see Figure 15). Since $K_9 - E^{1}_{11}, K_9 - E^{2}_{11} \notin {\mathcal{G}_{\mathcal{N}}}^{5}$ and $K_9 - E^{t_1}_{11} \notin {\mathcal{G}_{\mathcal{N}}}^{4}$, for $t_1 \in [3, 7]$ by Theorem 2.5, $K_9 - E^{t_2}_{11} \notin \mathcal{R}(2K_2, F_7)$, for $t_2 \in [1, 7]$. Thus, $\mathcal{F}^{11}_{R} = \emptyset$. Therefore, there are no graphs obtained by removing any eleven edges from K_9 that belong to $\mathcal{R}(2K_2, F_7)$.

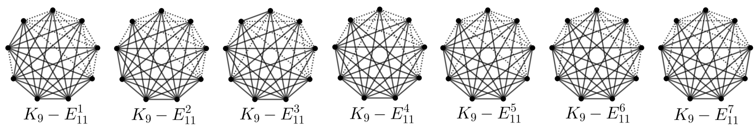

FIGURE 15. Graphs $K_9 - E^{i}_{11}$, for $i \in [1, 7]$

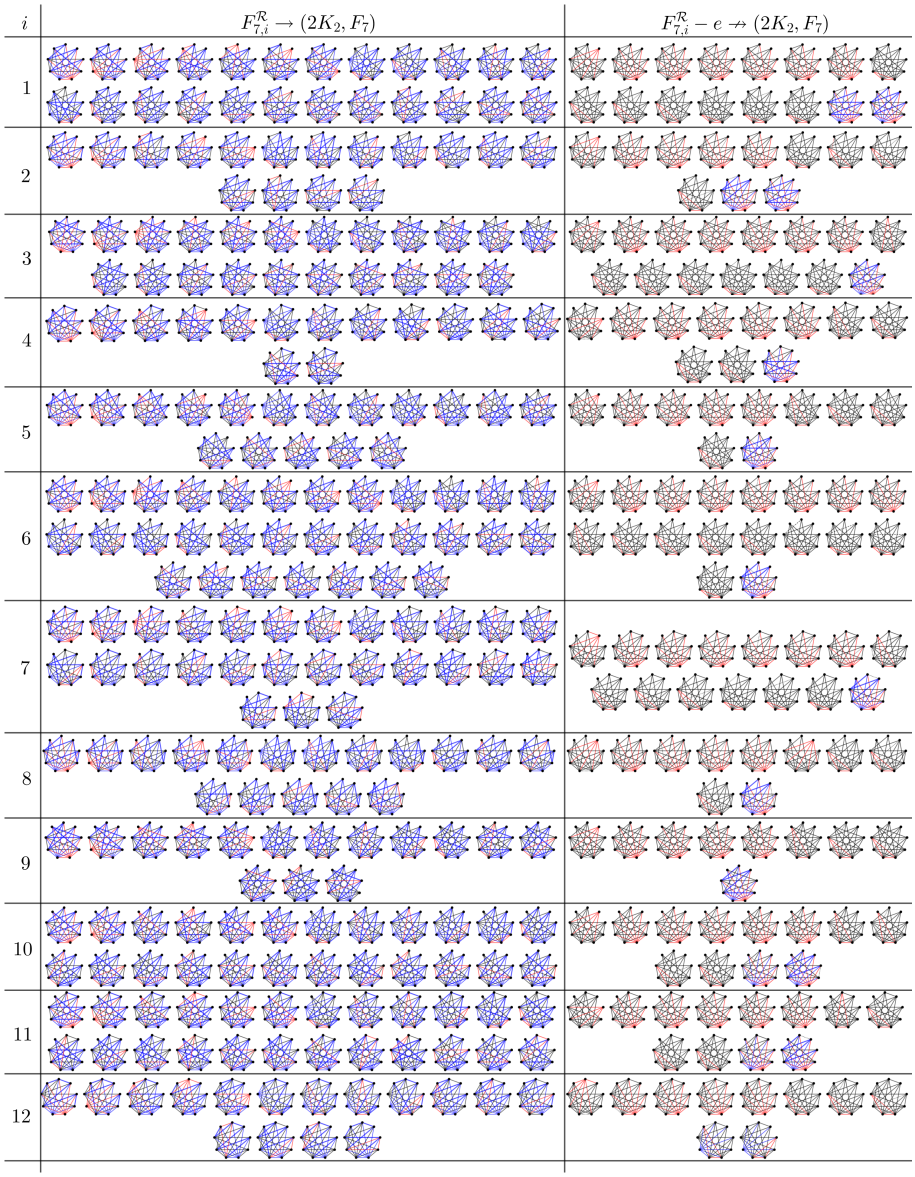

FIGURE 16. Red-Blue Coloring of Graphs $F^{\mathcal{R}}_{7,i}$ and $F^{\mathcal{R}}_{7,i} - e$, for $i \in [1, 12]$

Case 2. $K_9 - E_{10}$

In this case we explore the graphs $K_9 - E^j_{11} + e$ for $j \in [1, 7]$ and any edge e. We obtain 13 non-isomorphic graphs $F^{\mathcal{R}}_{7,i}$ and $K_9 - E^k_{10}$, for $i \in [1, 7]$ and $k \in [1, 6]$, as depicted in Figure 14 and 17. Since $K_9 - E^1_{10} \notin {\mathcal{G}_{\mathcal{N}}}^5$, $K_9 - E^{t_1}_{10} \notin {\mathcal{G}_{\mathcal{N}}}^4$, and $K_9 - E^5_{10}, K_9 - E^6_{10} \notin {\mathcal{G}_{\mathcal{N}}}^1$, for $t_1 \in [2, 4]$ by Theorem 2.5,

$K_9 - E_{10}^{t_2} \notin \mathcal{R}(2K_2, F_7)$, for $t_2 \in [1,6]$. Thus, $\mathcal{F}_R^{10} = \{F_{7,j}^{\mathcal{R}} | j \in [1,7]\}$. Therefore, removing any ten edges from the graph K_9 only produces the graphs $F_{7,j}^{\mathcal{R}}$ for $j \in [1,7]$ as members of $\mathcal{R}(2K_2, F_7)$.

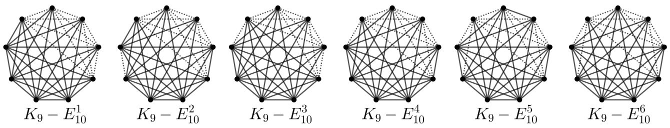

FIGURE 17. Graphs $K_9 - E_{10}^i$, for $i \in [1,6]$

Case 3. $K_9 - E_9$

In this case we explore the graphs $K_9 - E_{10}^j + e$ for $j \in [1,6]$ and any edge e. We obtain 16 non-isomorphic graphs $F_{7,i}^{\mathcal{R}}$ and $K_9 - E_9^k$, for $i \in [8,11]$ and $k \in [1,12]$, as depicted in Figure14 and 18. Since $K_9 - E_9^1 \notin {\mathcal{G}_{\mathcal{N}}}^4$, $K_9 - E_9^2, K_9 - E_9^3 \notin {\mathcal{G}_{\mathcal{N}}}^3$, and $K_9 - E_9^4, K_9 - E_9^5 \notin {\mathcal{G}_{\mathcal{N}}}^1$, by Theorem 2.5, $K_9 - E_9^{t_1} \notin \mathcal{R}(2K_2, F_7)$, for $t_1 \in [1,5]$. Since $K_9 - E_9^6 \supset F_{7,1}^{\mathcal{R}}$, $K_9 - E_9^7 \supset F_{7,4}^{\mathcal{R}}$, $K_9 - E_9^8 \supset F_{7,5}^{\mathcal{R}}$, $K_9 - E_9^9 \supset F_{7,7}^{\mathcal{R}}$, $K_9 - E_9^{10} \supset F_{7,8}^{\mathcal{R}}$, $K_9 - E_9^{11} \supset F_{7,11}^{\mathcal{R}}$, and $K_9 - E_9^{12} \supset F_{7,10}^{\mathcal{R}}$, then $K_9 - E_9^{t_1} \notin \mathcal{R}(2K_2, F_7)$ for $t_1 \in [6,12]$. Thus, $\mathcal{F}_R^9 = \{F_{7,j}^{\mathcal{R}} | j \in [8,11]\}$. Therefore, removing any nine edges from the graph K_9 only produces the graphs $F_{7,j}^{\mathcal{R}}$ for $j \in [8,11]$ as members of $\mathcal{R}(2K_2, F_7)$.

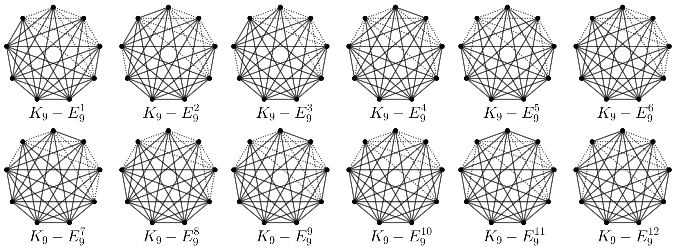

FIGURE 18. Graphs $K_9 - E_9^i$, for $i \in [1,12]$

Case 4. $K_9 - E_8$

In this case we explore the graphs $K_9 - E_9^j + e$ for $j \in [1,5]$ and any edge e. We obtain 15 non-isomorphic graphs and $F_{7,12}^{\mathcal{R}}$ and $K_9 - E_8^i$, for $i \in [1,14]$, as depicted in Figure 14 and 19. Since $K_9 - E_8^1, K_9 - E_8^2 \notin {\mathcal{G}_{\mathcal{N}}}^3$, and $K_9 - E_8^3, K_9 - E_8^4 \notin {\mathcal{G}_{\mathcal{N}}}^1$, by Theorem 2.5, $K_9 - E_8^{t_1} \notin \mathcal{R}(2K_2, F_7)$, for $t_1 \in [1,4]$. Since $K_9 - E_8^5 \supset F_{7,1}^{\mathcal{R}}$, $K_9 - E_8^6 \supset F_{7,2}^{\mathcal{R}}$, $K_9 - E_8^7 \supset F_{7,3}^{\mathcal{R}}$, $K_9 - E_8^8 \supset F_{7,7}^{\mathcal{R}}$, $K_9 - E_8^{11} \supset F_{7,8}^{\mathcal{R}}$, $K_9 - E_8^{t_2} \supset F_{7,10}^{\mathcal{R}}$, and $K_9 - E_8^{13} \supset F_{7,11}^{\mathcal{R}}$, for $t_2 \in \{9,10,12,14\}$ then $K_9 - E_8^{t_3} \notin \mathcal{R}(2K_2, F_7)$ for $t_3 \in [5,14]$. Thus, $\mathcal{F}_R^8 = \{F_{7,12}^{\mathcal{R}}\}$. Therefore, removing any eight edges from the graph K_9 only produces the graph $F_{7,12}^{\mathcal{R}}$ as member of $\mathcal{R}(2K_2, F_7)$.

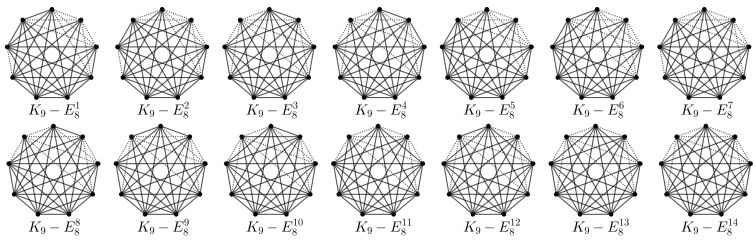

FIGURE 19. Graphs $K_9 - E_8^i$, for $i \in [1, 14]$

Case 5. $K_9 - E_7$

In this case we explore the graphs $K_9 - E_8^j + e$ for $j \in [1,4]$ and any edge e. We obtain 13 non-isomorphic graphs $K_9 - E_7^i$, for $i \in [1,13]$, as depicted in Figure 20. Since $K_9 - E_7^1 \notin \mathcal{G_N}^3$ and $K_9 - E_7^2 \notin \mathcal{G_N}^1$, by Theorem 2.5, $K_9 - E_7^{t_1} \notin \mathcal{R}(2K_2, F_7)$, for $t_1 \in [1,2]$. Since $K_9 - E_7^{t_2} \supset F_{7,1}^{\mathcal{R}}$, $K_9 - E_7^7 \supset F_{7,6}^{\mathcal{R}}$, $K_9 - E_7^{12} \supset F_{7,7}^{\mathcal{R}}$, $K_9 - E_7^{t_3} \supset F_{7,10}^{\mathcal{R}}$, $K_9 - E_7^9 \supset F_{7,11}^{\mathcal{R}}$, and $K_9 - E_7^{t_4} \supset F_{7,12}^{\mathcal{R}}$, for $t_2 \in \{3,6\}$, $t_3 \in \{10,11,13\}$, and $t_4 \in \{4,5,8\}$, then $K_9 - E_7^{t_5} \notin \mathcal{R}(2K_2, F_7)$ for $t_5 \in [3,13]$. Thus, $\mathcal{F}_R^7 = \emptyset$. Therefore, there are no graphs obtained by removing any seven edges from K_9 that belong to $\mathcal{R}(2K_2, F_7)$.

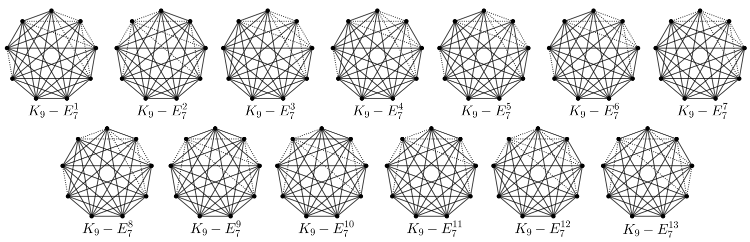

FIGURE 20. Graphs $K_9 - E_7^i$, for $i \in [1, 13]$

Case 6. $K_9 - E_6$

In this case we explore the graphs $K_9 - E_7^j + e$ for $j \in [1,2]$ and any edge e. We obtain five non-isomorphic graphs $K_9 - E_6^i$, for $i \in [1,5]$, as depicted in Figure 21. Since $K_9 - E_6^1 \notin \mathcal{G_N}^1$ by Theorem 2.5, $K_9 - E_6^1 \notin \mathcal{R}(2K_2, F_7)$. Since $K_9 - E_6^{t_1} \supset F_{7,1}^{\mathcal{R}}$, $K_9 - E_6^2 \supset F_{7,3}^{\mathcal{R}}$, and $K_9 - E_6^5 \supset F_{7,10}^{\mathcal{R}}$, for $t_1 \in \{3,4\}$, then $K_9 - E_6^{t_2} \notin \mathcal{R}(2K_2, F_7)$ for $t_3 \in [2,5]$. Thus, $\mathcal{F}_R^6 = \emptyset$. Therefore, there are no graphs obtained by removing any six edges from K_9 that belong to $\mathcal{R}(2K_2, F_7)$.

Case 7. $K_9 - E_k$, for $t \leq 5$.

Since $K_9 - E_6^1 + E_i \supset F_{7,10}^{\mathcal{R}}$, for $i \leq 6$ then $K_9 - E_6^1 + E_i \notin \mathcal{R}(2K_2, F_7)$. Thus, $\mathcal{F}_R^t = \emptyset$, for $t \leq 5$. Therefore, there are no graphs obtained from removing at most five edges from K_9 as a member of $\mathcal{R}(2K_2, F_7)$.

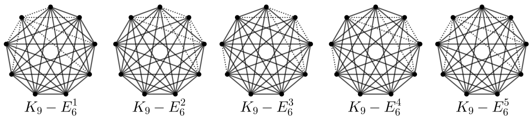

FIGURE 21. Graphs $K_9 - E_6^i$, for $i \in [1, 5]$

Based on Case $1 - 7$, the graphs with nine vertices and maximum degree 8 in $\mathcal{R}(2K_2, F_7)$ are only $F_{7,i}^{\mathcal{R}}$, for $i \in [1, 12]$. $\square$

3. CONCLUDING REMARKS

3.1. **Conclusion.**

In this research, we obtain the two main results as follows.

(1) By Theorem 2.3, we obtain $K_{(\frac{n}{2}+1)\times 2}$ is the unique graph in $\mathcal{R}(2K_2, F_n)$ with minimum order and maximum degree n, for $n \geq 4$.

(2) Let $\hat{\mathcal{R}}(2K_2, F_n)$ denote the set of all graphs of smallest order that are Ramsey $(2K_2, F_n)$-minimal. By Theorem 2.7-2.11, we obtain

$$\hat{\mathcal{R}}(2K_2, F_n) = \begin{cases} \{K_5\}, & \text{for } n = 3 \\ \{K_{3\times 2}\}, & \text{for } n = 4 \\ \{F_5^{\mathcal{R}}\}, & \text{for } n = 5 \\ \{K_{4\times 2}, F_{6,i}^{\mathcal{R}} \mid i \in [1, 5]\}, & \text{for } n = 6 \\ \{F_{7,i}^{\mathcal{R}} \mid i \in [1, 12]\}, & \text{for } n = 7. \end{cases}$$

3.2. **Open Problem.**

Pursuing this line of research further, identifying the Ramsey $(2K_2, F_n)$-minimal of the smallest order for $n \geq 8$ remains an unresolved issue. Readers are also encouraged to find a complete list of Ramsey $(2K_2, F_n)$-minimal for all orders for $n \geq 3$.

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REFERENCES

[1] F. P. Ramsey, "On a problem of formal logic," in *Proceedings of the London Mathematical Society*, vol. 30, pp. 263–286, 1930. `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/ramseyorig.pdf`.

[2] S. A. Burr, P. Erdos, and L. Lovász, "On graphs of ramsey type," *Ars Combinatoria*, vol. 1, pp. 167–190, 1976. `https://api.semanticscholar.org/CorpusID:17474857`.

[3] R. Diestel, *Graph Theory (5th. ed.)*. Springer Publishing Company, Incorporated, 2017. `https://doi.org/10.1007/978-3-662-53622-3`.

[4] E. T. Baskoro, S. M. Nababan, and M. Miller, "On ramsey numbers for trees versus wheels of five or six vertices," *Communications in Algebra*, vol. 18, no. 4, pp. 717–721, 2002. https://doi.org/10.1007/s003730200056.

[5] S. A. Burr, "Generalized ramsey theory for graphs–a survey," *Graphs and Combinatorics. Springer, Berlin*, vol. 406, pp. 52–75, 1973. https://doi.org/10.1007/BFb0066435.

[6] C. J. Jayawardene and C. C. Rousseau, "The ramsey number for a cycle of length five vs. a complete graph of order six," *J. Graph Theory*, vol. 35, pp. 99–108, 2000. https://doi.org/10.1002/1097-0118(200010)35:2<99::AID-JGT4>3.0.CO;2-6.

[7] Surahmat, E. T. Baskoro, and H. J. Broersma, "The ramsey numbers of large cycles versus small wheels," *INTEGERS Electron. J. Combin. Number Theory*, vol. 4, no. 10, pp. 1–9, 2004. https://www.kurims.kyoto-u.ac.jp/EMIS/journals/INTEGERS/papers/e10/e10.pdf.

[8] T. Łuczak, "On ramsey minimal graphs," *The Electronic Journal of Combinatorics*, vol. 1, p. #R4, 1994. https://doi.org/10.37236/1184.

[9] S. A. Burr, P. Erdős, R. J. Faudree, and R. H. Schelp, "A class of ramsey-finite graphs," in *Proceeding of the Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing*, pp. 171–180, 1978. https://static.renyi.hu/~p_erdos/1978-02.pdf.

[10] E. T. Baskoro, T. Vetrík, and L. Yulianti, "A note on ramsey $(K_{1,2}, C_4)$-minimal graphs of diameter 2," in *Proceedings of the International Conference 70 Years of FCE STUs*, Bratislava, Slovakia, 2008.

[11] E. T. Baskoro, L. Yulianti, and A. H., "Ramsey $(K_{1,2}, C_4)$-minimal graphs," *J. Combin. Math. Combin. Comput.*, vol. 65, pp. 79–90, 2008.

[12] T. Vetrík, L. Yulianti, and B. E. T., "On ramsey $(K_{1,2}, C_4)$-minimal graphs," *Discuss. Math. Graph Theory*, vol. 30, p. 637–649, 2010. https://doi.org/10.7151/dmgt.1519.

[13] L. Yulianti, H. Assiyatun, S. Uttunggadewa, and B. E. T., "On ramsey $(K_{1,2}, P_4)$-minimal graphs," *Far East J. Math. Sci.*, vol. 40, no. 1, pp. 23–26, 2010.

[14] M. Borowiecki, I. Schiermeyer, and E. Sidorowicz, "Ramsey $(K_{1,2}, K_3)$-minimal graphs," *Electron. J. Combin.*, vol. 12, no. R20, pp. 1–15, 2005. https://www.combinatorics.org/ojs/index.php/eljc/article/view/v12i1r20/pdf.

[15] M. Borowiecka-Olszewska and M. Haluszckak, "On ramsey $(K_{1,m}, G)$-minimal graphs," *Discrete Math*, vol. 313, no. 19, p. 1843–1855, 2012. https://doi.org/10.1016/j.disc.2012.06.020.

[16] I. Mengersen and J. Oeckermann, ""matching-star ramsey sets," *Discrete Appl. Math.*, vol. 95, pp. 417–424, 1999. https://doi.org/10.1016/S0166-218X(99)00089-X.

[17] H. Muhshi and E. T. Baskoro, "On ramsey $(3K_2, P_3)$-minimal graphs," in *AIP Conf. Proc.*, vol. 1450, pp. 110–117, 2012. https://doi.org/10.1063/1.4940826.

[18] E. T. Baskoro and L. Yulianti, "On ramsey minimal graphs for $2K_2$ versus P_n," *Adv. Appl. Discrete Math.*, vol. 8, no. 2, pp. 83–90, 2011.

[19] D. Tatanto and B. E. T., "On ramsey $(2K_2, 2P_n)$-minimal graphs," in *AIP Conference Proceedings*, vol. 1450, pp. 90–95, 2012. https://doi.org/10.1063/1.4724122.

[20] D. R. Silaban, A. I. Taufiq, and K. Wijaya, "On ramsey (mK_2, P_4)-minimal graphs," *SIAM Journal on Discrete Mathematics*, vol. 2, p. 467–488, 2008. https://doi.org/10.2991/acsr.k.220202.003.

[21] K. Wijaya, E. T. Baskoro, H. Assiyatun, and D. Suprijanto, "The complete list of ramsey $(2K_2, K_4)$-minimal graphs," *Electron. J. Graph Theory Appl.*, vol. 3, no. 2, pp. 216–227, 2015. http://dx.doi.org/10.5614/ejgta.2015.3.2.9.

[22] K. Wijaya, E. T. Baskoro, H. Assiyatun, and D. Suprijanto, "On unicyclic ramsey (mK_2, P_3)-minimal graphs," *Proc. Graph Theory*, vol. 74, pp. 10–14, 2015. https://doi.org/10.1016/j.procs.2015.12.067.

[23] E. T. Baskoro and W. K., "On ramsey $(2K_2, K_4)$-minimal graphs," *Mathematics in the 21st Century, Springer Proc. Math. Stat.*, vol. 98, pp. 11–17, 2015. https://doi.org/10.1007/978-3-0348-0859-0_2.

[24] K. Wijaya, E. T. Baskoro, H. Assiyatun, and D. Suprijanto, "On ramsey (mK_2, H)-minimal graphs," *Graphs Comb.*, vol. 23, pp. 233–243, 2017. https://doi.org/10.1007/s00373-016-1748-1.

[25] M. R. Fajri, A. H., and E. T. Baskoro, "On ramsey $(2K_2, W_n)$-minimal graphs of smallest order," *Electronic Journal of Graph Theory and Applications*, 2024. (On Review).

[26] M. R. Fajri, A. H., and E. T. Baskoro, "The characterization of ramsey $(2K_2, Fr_n)$-minimal graphs of smallest order," in *AIP Conf. Proc.*, 2025. (Accepted).