NEW SUBCLASS OF UNIVALENT FUNCTIONS DEFINED BY USING GENERALIZED SALAGEAN OPERATOR

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Abstract. In this paper, we have introduced and studied a new subclass $T_{\lambda}(\alpha, \beta, \xi; n)$ of univalent functions defined by using generalized Salagean operator in the unit disk $U = \{z : |z| < 1\}$ We have obtained among others results like, coefficient inequalities, distortion theorem, extreme points, neighbourhood and Hadamard product properties.

1. INTRODUCTION

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. In [4], Al-oboudi defined a differential operator as follows, for a function $f \in A$,

$$D^0 f(z) = f(z),$$
$$D f(z) = D^1 f(z) = (1 - \lambda)f(z) + \lambda zf'(z) = D_\lambda f(z), \lambda \geq 0,$$

in general

$$D^n f(z) = D_\lambda(D^{n-1} f(z)).$$

If $f(z)$ is given by (1), then from (2) and (3) we observe that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k - 1)\lambda]^n a_k z^k$$
when $\lambda = 1$, get Salagean differential operator [7]. Further, let $T$ denote the subclass of $A$ which consists of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0.$$  \hfill (5)

A function $f(z)$ belonging to $A$ is in the class $D_\lambda(\alpha, \beta; n)$, if and only if

$$\left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - 1 \right| < 2\xi \left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - \alpha \right| < \beta$$  \hfill (6)

where $0 \leq \alpha < \frac{1}{2}\xi$, $0 < \beta \leq 1$, $1/2 \leq \xi \leq 1$, $n \in N \cup \{0\}$, $z \in U$. Let

$$TD_\lambda(\alpha, \beta; n) = T \cap D_\lambda(\alpha, \beta; n).$$  \hfill (7)

2. MAIN RESULTS

**Theorem 1.** Let $f$ be defined by (5). Then $f \in TD_\lambda(\alpha, \beta; n)$ if and only if

$$\sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k [1 + \beta(2\xi - 1)]a_k \leq 2\beta\xi(1 - \alpha)$$  \hfill (8)

$0 \leq \alpha < \frac{1}{2}\xi$, $0 < \beta \leq 1$, $1/2 \leq \xi \leq 1$, $n \in N \cup \{0\}$, $\lambda \geq 0$.

**Proof.** For $|z| = 1$, we get

$$\left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - 1 - 2\xi \left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - \alpha \right| - \left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - 1 \right| \right| = \left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - 1 \right|$$

$$= -\sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k a_k z^{k-1} - \beta \left[ 2\xi(1 - \alpha) - 2\xi \sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k a_k z^{k-1} + \sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k a_k z^{k-1} \right]$$

$$\leq \sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k a_k z^{k-1} - 2\beta\xi(1 - \alpha) \leq 0,$$

by hypothesis. Thus by maximum modulus theorem, we have $f \in TD_\lambda(\alpha, \beta; n)$.

Conversely, suppose that $f \in TD_\lambda(\alpha, \beta; n)$ hence the condition (6) gives us

$$\left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - 1 \right| = \left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - \alpha \right| - \left| \left( \frac{\partial^n f(z)}{\partial z^n} \right)' - 1 \right|$$

$$\leq \sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k a_k z^{k-1}$$

$$\leq 2\xi(1 - \alpha) - (2\xi - 1) \sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right]^{n-1} k a_k z^{k-1}$$

$$< \beta.$$
Since $|\text{Re}(z)| < |z|$ for all $z$, we obtain

$$
\text{Re} \left\{ -\sum_{k=2}^{\infty} |1 + (k - 1)\lambda|^n k a_k z^{k-1} \right\} < \beta.
$$

Letting $z \to 1^-$ through real values, we get (8). The result is sharp for the function

$$
f(z) = \frac{2\beta \xi (1 - \alpha)}{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]} z^k, k \geq 2.
$$

**Corollary 1.** Let $f \in T$ belong to the class $TD_{\lambda}(\alpha, \beta, \xi; n)$ then

$$
a_k \leq \frac{2\beta \xi (1 - \alpha)}{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}, k \geq 2.
$$

**Theorem 2.** Let $f \in T$ belong to the class $TD_{\lambda}(\alpha, \beta, \xi; n)$, then for $|z| \leq r < 1$, we have

$$
r - r^2 \frac{\beta \xi (1 - \alpha)}{1 + \beta(2\xi - 1)} \leq |D^n f(z)| \leq r + r^2 \frac{\beta \xi (1 - \alpha)}{1 + \beta(2\xi - 1)}
$$

and

$$
1 - r \frac{2\beta \xi (1 - \alpha)}{1 + \beta(2\xi - 1)} \leq |(D^n f(z))'| \leq 1 + r \frac{2\beta \xi (1 - \alpha)}{1 + \beta(2\xi - 1)}.
$$

Bounds given by (10) and (11) are sharp.

**Proof.** By Theorem 1, we have

$$
\sum_{k=2}^{\infty} |1 + (k - 1)\lambda|^n k[1 + \beta(2\xi - 1)]a_k \leq 2\beta \xi (1 - \alpha)
$$

then, we have

$$
2(1 + \lambda)^n[1 + \beta(2\xi - 1)]a_k \leq \sum_{k=2}^{\infty} |1 + (k - 1)\lambda|^n k[1 + \beta(2\xi - 1)]a_k \leq 2\beta \xi (1 - \alpha),
$$

then,

$$
\sum_{k=2}^{\infty} a_k \leq \frac{2\beta \xi (1 - \alpha)}{2(1 + \lambda)^n[1 + \beta(2\xi - 1)]}.
$$
Hence

\[ |D^n f(z)| \leq |z| + \sum_{k=2}^{\infty} |[1 + (k - 1)\lambda]^n a_k z^k| \]
\[ \leq |z| + |z|^2 (1 + \lambda)^n \sum_{k=2}^{\infty} a_k \]
\[ \leq r + r^2 (1 + \lambda)^n \sum_{k=2}^{\infty} a_k \]
\[ \leq r + r^2 \frac{\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)}, \]

and

\[ |D^n f(z)| \geq |z| - \sum_{k=2}^{\infty} |[1 + (k - 1)\lambda]^n a_k z^k| \]
\[ \geq |z| - |z|^2 (1 + \lambda)^n \sum_{k=2}^{\infty} a_k \]
\[ \geq r - r^2 (1 + \lambda)^n \sum_{k=2}^{\infty} a_k \]
\[ \geq r - r^2 \frac{\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)}, \]

thus (10) is true. Further,

\[ |(D^n f(z))'| \leq 1 + 2r (1 + \lambda)^n \sum_{k=2}^{\infty} a_k \]
\[ \leq 1 + r \frac{2\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)}, \]

and

\[ |(D^n f(z))'| \geq 1 - 2r (1 + \lambda)^n \sum_{k=2}^{\infty} a_k \]
\[ \geq 1 - r \frac{2\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)}. \]

The result is sharp for the function \( f(z) \) defined by

\[ f(z) = z - \frac{2\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)} z^2, \quad z = \pm r. \]
Theorem 3. Let \( n \in \mathbb{N} \cup \{0\} \), \( \lambda \geq 0 \), \( 0 \leq \alpha_1 \leq \alpha_2 < \frac{1}{2} \xi \), \( 0 < \beta \leq 1 \), \( 1/2 \leq \xi \leq 1 \). Then \( TD_{\lambda}(\alpha_2, \beta, \xi; n) \subset TD_{\lambda}(\alpha_1, \beta, \xi; n) \).

Proof. By assumption we have

\[
\frac{2\beta(1-\alpha_2)}{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]} \leq \frac{2\beta(1-\alpha_1)}{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}
\]

Thus, \( f(z) \in TD_{\lambda}(\alpha_2, \beta, \xi; n) \) implies that

\[
\sum_{k=2}^{\infty} [1+(k-1)\lambda]^n a_k \leq \frac{2\beta(1-\alpha_2)}{[1+\beta(2\xi-1)]} \leq \frac{2\beta(1-\alpha_1)}{[1+\beta(2\xi-1)]} \leq \sum_{k=2}^{\infty} [1+(k-1)\lambda]^n a_k
\]

then \( f(z) \in TD_{\lambda}(\alpha_1, \beta, \xi; n) \).

Theorem 4. The set \( TD_{\lambda}(\alpha, \beta, \xi; n) \) is the convex set.

Proof. Let \( f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i}z^k \) (\( i = 1, 2 \)) belong to \( TD_{\lambda}(\alpha, \beta, \xi; n) \) and let \( g(z) = \zeta_1 f_1(z) + \zeta_2 f_2(z) \) with \( \zeta_1 \) and \( \zeta_2 \) non negative and \( \zeta_1 + \zeta_2 = 1 \), we can write

\[
g(z) = z - \sum_{k=2}^{\infty} (\zeta_1 a_{k,1} + \zeta_2 a_{k,2})z^k
\]

It is sufficient to show that \( g(z) = TD_{\lambda}(\alpha, \beta, \xi; n) \) that means

\[
\sum_{k=2}^{\infty} [1+(k-1)\lambda]^n k[1+\beta(2\xi-1)](\zeta_1 a_{k,1} + \zeta_2 a_{k,2}) \leq \zeta_1 (2\beta(1-\alpha)) + \zeta_2 (2\beta(1-\alpha)) = (\zeta_1 + \zeta_2)(2\beta(1-\alpha)) = 2\beta(1-\alpha)
\]

Thus \( g(z) \in TD_{\lambda}(\alpha, \beta, \xi; n) \).

We shall now present a result on extreme points in the following theorem

Theorem 5. Let \( f_1(z) = z \) and

\[
f(z) = z - \frac{2\beta(1-\alpha)}{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]} z^k
\]
for all \( k \geq 2, n \in \mathbb{N} \cup \{0\}, \lambda \geq 0, 0 \leq \alpha < \frac{1}{2}\xi, 0 < \beta \leq 1, 1/2 \leq \xi \leq 1. \) Then \( f(z) \) is in the subclass \( TD_\lambda(\alpha, \beta, \xi; n) \) if and only if it can be expressed in the form \( f(z) = \sum_{k=2}^{\infty} \gamma_k z^k \) where \( \gamma_k \geq 0 \) and \( \sum_{k=2}^{\infty} \gamma_k = 1 \) or \( 1 = \gamma_1 + \sum_{k=2}^{\infty} \gamma_k. \)

**Proof.** Let \( f(z) = \sum_{k=2}^{\infty} \gamma_k z^k \) where \( \gamma_k \geq 0 \) and \( \sum_{k=2}^{\infty} \gamma_k = 1. \) Thus \( f(z) = z - \sum_{k=2}^{\infty} \frac{2\beta \xi (1 - \alpha)}{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]} \gamma_k z^k \) and we obtain

\[
\sum_{k=2}^{\infty} \left( \frac{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}{2\beta \xi (1 - \alpha)} \gamma_k \times \frac{2\beta \xi (1 - \alpha)}{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]} \right) = \sum_{k=2}^{\infty} \gamma_k = 1 - \gamma_1 \leq 1.
\]

In view of Theorem 1, this show that \( f(z) \in TD_\lambda(\alpha, \beta, \xi; n). \)

Conversely, suppose that of the form (5) belong to \( TD_\lambda(\alpha, \beta, \xi; n) \) then

\[
a_k \leq \frac{2\beta \xi (1 - \alpha)}{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}, \quad k \geq 2,
\]

Putting \( \gamma_k = \frac{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}{2\beta \xi (1 - \alpha)} \) and \( \gamma_1 = 1 - \sum_{k=2}^{\infty} \gamma_k, \) then we have \( f(z) = \gamma_1 f_1(z) + \sum_{k=2}^{\infty} \gamma_k f_k(z). \) This completes the proof.

### 3. Neighbourhood and Hadamard Product Properties

**Definition 1.** [6] Let \( \gamma_k \geq 0 \) and \( f(z) \in T \) of the form (5). The \((k, \gamma)-\)neighbourhood of a function \( f(z) \) defined by

\[
N_{(k, \gamma)}(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \leq \gamma \right\}
\]

For the identity function \( e(z) = z \) we have

\[
N_{(k, \gamma)}(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|b_k| \leq \gamma \right\}.
\]
Theorem 5. Let

\[ \gamma = \frac{2\beta \xi (1 - \alpha)}{(1 + \lambda)^n [1 + \beta (2\xi - 1)]}. \]

Then \( TD_\lambda(\alpha, \beta, \xi; n) \subset N_{(k, \gamma)}(e) \).

Proof. Let \( f \in TD_\lambda(\alpha, \beta, \xi; n) \) then we have

\[
2(1 + \lambda)^n [1 + \beta (2\xi - 1)] \sum_{k=2}^{\infty} a_k \\
\leq \sum_{k=2}^{\infty} [1 + (k - 1)\lambda]^{-n} k [1 + \beta (2\xi - 1)] a_k \\
\leq 2\beta \xi (1 - \alpha),
\]

therefore

\[
\sum_{k=2}^{\infty} a_k \leq \frac{\beta \xi (1 - \alpha)}{(1 + \lambda)^n [1 + \beta (2\xi - 1)]},
\]

(14)

also we have for \(|z| < r\)

\[
|f'(z)| \leq 1 + |z| \sum_{k=2}^{\infty} ka_k \leq 1 + r \sum_{k=2}^{\infty} ka_k.
\]

In view (14), we have

\[
|f'(z)| \leq 1 + r \frac{2\beta \xi (1 - \alpha)}{(1 + \lambda)^n [1 + \beta (2\xi - 1)]}.
\]

From above inequalities we get

\[
\sum_{k=2}^{\infty} ka_k \leq \frac{2\beta \xi (1 - \alpha)}{(1 + \lambda)^n [1 + \beta (2\xi - 1)]} = \gamma,
\]

therefore \( f \in N_{(k, \gamma)}(e) \).

Definition 2. The function \( f(z) \) defined by (5) is said to be a member of the subclass \( TD_\lambda(\alpha, \beta, \xi, \zeta; n) \) if there exits a function \( g \in TD_\lambda(\alpha, \beta, \xi; n) \) such that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| \leq 1 - \zeta, z \in U, 0 \leq \zeta < 1.
\]
**Theorem 6.** Let $g \in TD_{\lambda}(\alpha, \beta, \xi; n)$ and

$$\zeta = 1 - \frac{\gamma}{2} d(\alpha, \beta, \xi; n).$$

Then $N_{(k,\gamma)}(g) \subset TD_{\lambda}(\alpha, \beta, \xi, \zeta; n)$ where $n \in \mathbb{N} \cup \{0\}$, $\lambda \geq 0$, $0 \leq \alpha < \frac{1}{2}\xi$, $0 \leq \beta \leq \frac{1}{2}$, $\frac{1}{2} \leq \xi \leq 1$, $0 \leq \zeta < 1$ and

$$d(\alpha, \beta, \xi; n) = \frac{(1 + \lambda)^n[1 + \beta(2\xi - 1)]}{(1 + \lambda)^n[1 + \beta(2\xi - 1)] - \beta(1 - \alpha)}.$$

**Proof.** Let $f \in N_{(k,\gamma)}(g)$ then by (14) we have

$$\sum_{k=2}^{\infty} k|a_k - b_k| \leq \gamma,$$ then $\sum_{k=2}^{\infty} |a_k - b_k| \leq \gamma$. Since $g \in TD_{\lambda}(\alpha, \beta, \xi; n)$ we have

$$\sum_{k=2}^{\infty} b_k \leq \frac{\beta(1 - \alpha)}{(1 + \lambda)^n[1 + \beta(2\xi - 1)]},$$

therefore,

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{k=2}^{\infty} |a_k - b_k|}{1 - \sum_{k=2}^{\infty} b_k} \leq \frac{\gamma}{2} \left( \frac{(1 + \lambda)^n[1 + \beta(2\xi - 1)]}{(1 + \lambda)^n[1 + \beta(2\xi - 1)] - \beta(1 - \alpha)} \right) = \frac{\gamma}{2} d(\alpha, \beta, \xi; n) = 1 - \zeta.$$

Then by Definition 2, we get $f \in TD_{\lambda}(\alpha, \beta, \xi, \zeta; n)$.

**Theorem 7.** Let $f(z)$ and $g(z) \in TD_{\lambda}(\alpha_1, \beta, \xi; n)$ be of the form (5) such that $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ where $a_k, b_k \geq 0$. Then the Hadamard product $h(z)$ defined by $h(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k$ is in the subclass $TD_{\lambda}(\alpha_2, \beta, \xi; n)$ where

$$\alpha_2 \leq \frac{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)] - 2\beta(1 - \alpha_1)^2}{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}.$$

**Proof.** By Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}{2\beta(1 - \alpha_1)} a_k \leq 1$$

and

$$\sum_{k=2}^{\infty} \frac{[1 + (k - 1)\lambda]^n k[1 + \beta(2\xi - 1)]}{2\beta(1 - \alpha_1)} b_k \leq 1.$$
We have only to find the largest $\alpha_2$ such that

$$\sum_{k=2}^{\infty} \frac{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]}{2\beta\xi(1 - \alpha_2)} a_k b_k \leq 1.$$ 

Now, by Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty} \frac{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]}{2\beta\xi(1 - \alpha_1)} \sqrt{a_k b_k} \leq 1,$$

we need only to show that

$$\frac{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]}{2\beta\xi(1 - \alpha_2)} a_k b_k \leq \frac{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]}{2\beta\xi(1 - \alpha_1)} \sqrt{a_k b_k}, \quad (18)$$

equivalently,

$$\sqrt{a_k b_k} \leq \frac{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]}{2\beta\xi(1 - \alpha_1)} \times \frac{2\beta(1 - \alpha_2)}{[1 + (k - 1)\lambda]^n k [1 + \beta(2\zeta - 1)]} \leq \frac{1 - \alpha_2}{1 - \alpha_1}.$$

But from (18), we have

$$\sqrt{a_k b_k} \leq \frac{2\beta(1 - \alpha_1)}{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]}.$$

Consequently, we need to prove that

$$\frac{2\beta(1 - \alpha_1)}{(1 + (k - 1)\lambda)^n k [1 + \beta(2\zeta - 1)]} \leq \frac{1 - \alpha_2}{1 - \alpha_1},$$

or equivalently, that

$$\alpha_2 \leq \frac{[1 + (k - 1)\lambda]^n k [1 + \beta(2\zeta - 1)] - 2\beta(1 - \alpha_1)^2}{[1 + (k - 1)\lambda]^n k [1 + \beta(2\zeta - 1)]}.$$ 

\textbf{Theorem 8.} Let $f \in TD_\lambda(\alpha, \beta, \xi; n)$ be defined by (5) and $c$ any real number with $c > -1$ than the function $G(z)$ defined as $G(z) = \frac{c+1}{c} \int_0^z s^{c-1} f(s) \, ds$, $c > -1$, also belongs to $TD_\lambda(\alpha, \beta, \xi; n)$. 

Proof. By virtue of $G(z)$ it follows from (5) that

$$G(z) = \frac{c+1}{s^c} \int_0^z \left(s^c - \sum_{k=2}^{\infty} a_k s^{k+c-1}\right) ds$$

$$= z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) a_k z^k.$$

But

$$\sum_{k=2}^{\infty} \left[1 + (k-1)\lambda a_k [1 + \beta(2\xi - 1)] \left(\frac{c+1}{c+k}\right) a_k \right] \leq 1,$$

since $\left(\frac{c+1}{c+k}\right) \leq 1$ and by Theorem 1, so the proof is complete.

**Theorem 8.** Let $f \in TD_\lambda(\alpha, \beta, \xi; n)$ be defined by (5) and

$$F_\mu(z) = (1 - \mu)z + \mu \int_0^z \frac{f(s)}{s} ds \quad (\mu \geq 0, z \in U).$$

Then $F_\mu(z)$ is also in $TD_\lambda(\alpha, \beta, \xi; n)$ if $0 \leq \mu \leq 2$.

**Proof.** Let $f$ defined by (5) then

$$F_\mu(z) = (1 - \mu)z + \mu \int_0^z \left(s - \sum_{k=2}^{\infty} a_k s^k\right) ds$$

$$= z - \sum_{k=2}^{\infty} \frac{\mu}{k} a_k z^k.$$

By Theorem 1 and since $\left(\frac{\mu}{k}\right) \leq 1$ we have

$$\sum_{k=2}^{\infty} \left[1 + (k-1)\lambda a_k [1 + \beta(2\xi - 1)] \left(\frac{\mu}{k}\right) a_k \right] \leq \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda a_k [1 + \beta(2\xi - 1)] \left(\frac{\mu}{2}\right) a_k \right] \leq 1,$$

then $F_\mu(z)$ is in $TD_\lambda(\alpha, \beta, \xi; n)$.

**REFERENCES**


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