A NOTE ON INCLUSION PROPERTIES OF WEIGHTED ORLICZ SPACES

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Abstract. In this paper we present sufficient and necessary conditions for the inclusion relation between two weighted Orlicz spaces which complete the Osançılı result in 2014. One of the keys to prove our results is to use the norm of the characteristic functions of the balls in $\mathbb{R}^n$.

Key words and phrases: Inclusion property, weighted Lebesgue spaces, weighted Orlicz spaces.

1. INTRODUCTION

Orlicz spaces are generalization of Lebesgue spaces which were firstly introduced by Z. W. Birnbaum and W. Orlicz in 1931 (see [5, 14]). Let us first recall the definition of Orlicz spaces. Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function [that is, $\Phi$ is convex, $\lim_{t \to 0} \Phi(t) = 0 = \Phi(0)$, left-continuous and $\lim_{t \to \infty} \Phi(t) = \infty$]. The Orlicz space $L_\Phi(\mathbb{R}^n)$ is the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} \Phi(|f(x)|) dx < \infty$ for some $a > 0$. The space $L_\Phi(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|f\|_{L_\Phi(\mathbb{R}^n)} := \inf \left\{ b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \leq 1 \right\}.$$

Meanwhile, for $\Phi$ is a Young function, the weak Orlicz space $wL_\Phi(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|_{wL_\Phi(\mathbb{R}^n)} := \inf \left\{ b > 0 : \sup_{t > 0} \Phi(t) \mu \left( \left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\} \right) \leq 1 \right\} < \infty.$$
Now, we move to the weighted Orlicz spaces and weighted weak Orlicz spaces. Let $\Phi$ be a Young function and $u$ is a weight on $\mathbb{R}^n$ (i.e. $u : \mathbb{R}^n \to (0, \infty)$ is a measurable function). The **weighted Orlicz space** $L^\Phi_u(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $uf \in L^\Phi(\mathbb{R}^n)$. Note that, the space $L^\Phi_u(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$
\|f\|_{L^\Phi_u(\mathbb{R}^n)} := \|uf\|_{L^\Phi(\mathbb{R}^n)} = \inf \left\{ b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|uf(x)|}{b}\right) \, dx \leq 1 \right\}.
$$

Similar with weighted Orlicz spaces, for a Young function $\Phi$ and a weight $u$ on $\mathbb{R}^n$, the **weighted weak Orlicz space** $wL^\Phi_u(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{wL^\Phi_u(\mathbb{R}^n)} := \|uf\|_{wL^\Phi(\mathbb{R}^n)} < \infty$.

For $u_1, u_2 : \mathbb{R}^n \to (0, \infty)$, we denote $u_1 \preceq u_2$ if there exists a constant $C > 0$ such that $u_1(x) \leq C u_2(x)$ for all $x \in \mathbb{R}^n$. Note that, if $u_1 \preceq u_2$, then $\|f\|_{L^{\Phi_1}_u(\mathbb{R}^n)} \leq C \|f\|_{L^{\Phi_2}_u(\mathbb{R}^n)}$ and $\|f\|_{wL^{\Phi_1}_u(\mathbb{R}^n)} \leq C \|f\|_{wL^{\Phi_2}_u(\mathbb{R}^n)}$.

The study of Lebesgue spaces and Orlicz spaces has been carried out by many researchers in the last few decades (see [3, 5, 6, 7, 12, 15], etc.). In 1989, Maligranda [6] discussed inclusion properties of Orlicz spaces. Later in 2016, Masta et al. [7] obtained sufficient and necessary conditions for the inclusion relation between two Orlicz spaces and between two weak Orlicz spaces by using a different technique from Maligranda. Moreover, they have found that two Orlicz spaces and two weak Orlicz spaces are comparable with respect to Young functions for any measurable set, although the Lebesgue space $L_p$ are not comparable with respect to the number $p$.

On the other hand, Osançılıol [13] has obtained sufficient and necessary conditions for the inclusion relation between two weighted Orlicz spaces, as in the following theorem.

**Theorem 1.1.** [13] Let $\Phi$ be a continuous Young function satisfying the $\Delta_2$ condition [that is, there exists $K > 0$ such that $\Phi(2t) \leq K \Phi(t)$ for all $t \geq 0$], and $u_1, u_2$ are measurable functions such that $u_i(x + y) \leq u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1, 2$. Then the following statements are equivalent:

1. $u_1 \preceq u_2$.
2. $L^{\Phi_2}_u(\mathbb{R}^n) \subseteq L^{\Phi_1}_u(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that $\|f\|_{L^{\Phi_1}_{u_1}(\mathbb{R}^n)} \leq C \|f\|_{L^{\Phi_2}_{u_2}(\mathbb{R}^n)}$, for every $f \in L^{\Phi_2}_{u_2}(\mathbb{R}^n)$.

Related results for weak type of Orlicz spaces can be found in [10].

In this paper, we are interested in studying the inclusion properties of weighted Orlicz spaces. In connection with Theorem 1.1, we shall prove the inclusion relation between weighted Orlicz spaces with respect to Young functions $\Phi_1, \Phi_2$ and weights $u_1, u_2$.

To achieve our purpose, we will use the similar methods as in [1, 7, 8, 9, 13], which pay attention to the characteristic functions of open balls in $\mathbb{R}^n$. Next, we recall some lemmas which will be used later in the next section.
Lemma 1.2. [11] Suppose that \( \Phi \) is a Young function and \( \Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\} \). We have

1. \( \Phi^{-1}(0) = 0 \).
2. \( \Phi^{-1}(s_1) \leq \Phi^{-1}(s_2) \) for \( s_1 \leq s_2 \).
3. \( \Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s)) \) for \( 0 \leq s < \infty \).

Lemma 1.3. [9] Let \( \Phi_1, \Phi_2 \) be Young functions. For any \( s > 0 \), if there exists \( C_1, C_2 > 0 \) such that \( \Phi^{-1}_2(s) \leq C_1 \Phi^{-1}_1(C_2 s) \), then we have \( \Phi_1(\frac{1}{C_2}) \leq C_2 \Phi_2(t) \) for \( t = \Phi_2^{-1}(s) \).

In this paper, the letter \( C \) will be used for constants whose values may change from line to line, while constants with subscripts, such as \( C_1, C_2 \), do not change their values.

2. RESULTS

First, we will investigate the inclusion properties of weighted Orlicz spaces with respect to distinct Young functions \( \Phi_1 \) and \( \Phi_2 \). To get the result, we need to estimate the norm of the characteristic function of an open ball in \( \mathbb{R}^n \) as in the following lemma.

Lemma 2.1. [4, 7] Let \( \Phi \) be a Young function, \( a \in \mathbb{R}^n \), and \( r > 0 \) be arbitrary. Then we have \( \|\chi_{B(a,r)}\|_{L^n_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)} \) where \( |B(a,r)| \) denotes the volume of the open ball \( B(a,r) \) centered at \( a \in \mathbb{R}^n \) with radius \( r > 0 \).

Now we come to the inclusion relation between \( L^n_{\Phi_1}(\mathbb{R}^n) \) and \( L^n_{\Phi_2}(\mathbb{R}^n) \) with respect to Young functions \( \Phi_1, \Phi_2 \). Given two Young functions \( \Phi_1, \Phi_2 \), we write \( \Phi_1 \preceq \Phi_2 \) if there exists a constant \( C > 0 \) such that \( \Phi_1(t) \leq \Phi_2(Ct) \) for all \( t > 0 \).

Theorem 2.2. Let \( \Phi_1, \Phi_2 \) be Young functions and \( u : \mathbb{R}^n \to (0, \infty) \) be a measurable function. Then the following statements are equivalent:

1. \( \Phi_1 \preceq \Phi_2 \).
2. \( L^n_{\Phi_1}(\mathbb{R}^n) \subseteq L^n_{\Phi_2}(\mathbb{R}^n) \).
3. There exists a constant \( C > 0 \) such that \( \|f\|_{L^n_{\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{L^n_{\Phi_2}(\mathbb{R}^n)} \), for every \( f \in L^n_{\Phi_2}(\mathbb{R}^n) \).

Proof. Assume that (1) holds. Suppose that \( f \in L^n_{\Phi_2}(\mathbb{R}^n) \). Observe that

\[
\int_{\mathbb{R}^n} \Phi_1\left(\frac{|u(x)f(x)|}{C\|f\|_{L^n_{\Phi_2}(\mathbb{R}^n)}}\right) dx \leq \int_{\mathbb{R}^n} \Phi_2\left(\frac{C|u(x)f(x)|}{\|f\|_{L^n_{\Phi_2}(\mathbb{R}^n)}}\right) dx
= \int_{\mathbb{R}^n} \Phi_2\left(\frac{|u(x)f(x)|}{\|f\|_{L^n_{\Phi_2}(\mathbb{R}^n)}}\right) dx \leq 1.
\]
By definition of \( \| \cdot \|_{L^\Phi_2(\mathbb{R}^n)} \), we have \( \| f \|_{L^\Phi_2(\mathbb{R}^n)} \leq C \| f \|_{L^\Phi_1(\mathbb{R}^n)} \). This proves that \( L^u_{\Phi_2}(\mathbb{R}^n) \subseteq L^u_{\Phi_1}(\mathbb{R}^n) \).

Next, since \((L^u_{\Phi_1}(\mathbb{R}^n), L^u_{\Phi_2}(\mathbb{R}^n))\) is a Banach pair, it follows from [2, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1).

Assume now that (3) holds. By Lemma 2.1, we have

\[
\frac{1}{\Phi_1^{-1}\left(\frac{1}{\|B(a,r)\|}\right)} \leq \frac{\|\chi_{B(a,r)}\|_{L^\Phi_2(\mathbb{R}^n)}}{\|\chi_{B(a,r)}\|_{L^\Phi_1(\mathbb{R}^n)}} \leq C \frac{\|\chi_{B(a,r)}\|_{L^\Phi_2(\mathbb{R}^n)}}{\Phi_2^{-1}\left(\frac{1}{\|B(a,r)\|}\right)}.
\]

Since \( \frac{1}{\Phi_1^{-1}\left(\frac{1}{\|B(a,r)\|}\right)} \leq \frac{C}{\Phi_2^{-1}\left(\frac{1}{\|B(a,r)\|}\right)} \) is equivalent to \( C \Phi_1^{-1}\left(\frac{1}{\|B(a,r)\|}\right) \geq \Phi_2^{-1}\left(\frac{1}{\|B(a,r)\|}\right) \) for arbitrary \( a \in \mathbb{R}^n \) and \( r > 0 \), by Lemma 1.3, we have

\[
\Phi_1\left(\frac{t_0}{C}\right) \leq \Phi_2(t_0),
\]

for \( t_0 = \Phi_2^{-1}\left(\frac{1}{\|B(a,r)\|}\right) \). Since \( a \in \mathbb{R}^n \) and \( r > 0 \) are arbitrary, we conclude that \( \Phi_1(t) \leq \Phi_2(Ct) \) for every \( t > 0 \). \( \square \)

**Remark 2.3.** For \( u(x) = 1 \), Theorem 2.2 reduces to Theorem 2.5 in [7].

Next, we also give the sufficient and necessary conditions for inclusion relation between weighted Orlicz spaces \( L^u_{\Phi_1}(\mathbb{R}^n) \) and \( L^u_{\Phi_2}(\mathbb{R}^n) \) with respect to Young functions \( \Phi_1, \Phi_2 \) and weights \( u_1, u_2 \). To get the result, we need the following lemma.

**Lemma 2.4.** Let \( u : \mathbb{R}^n \to (0, \infty) \) be a measurable function such that \( u(x + y) \leq u(x) \cdot u(y) \) for every \( x, y \in \mathbb{R}^n \). If \( \Phi \) is a Young function, then:

1. For all \( f \in L^u_{\Phi}(\mathbb{R}^n) \) and for all \( x \in \mathbb{R}^n \), we have \( \|T_x f\|_{L^u_{\Phi}(\mathbb{R}^n)} \leq u(x) \|f\|_{L^u_{\Phi}(\mathbb{R}^n)} \), where \( T_x f = f(y - x) \).
2. If \( f \in L^u_{\Phi}(\mathbb{R}^n) \) and \( f \neq 0 \), then there exists a constant \( C > 0 \) (depending on \( f \)) such that

\[
\frac{u(x)}{C} \leq \|T_x f\|_{L^u_{\Phi}(\mathbb{R}^n)} \leq Cu(x).
\]

**Proof.**

1. Let \( f \in L^u_{\Phi}(\mathbb{R}^n) \) and \( T_x f(y) = f(y - x) \). Then

\[
\int_{\mathbb{R}^n} \Phi\left(\frac{|u(v)f(v)|}{\|f\|_{L^u_{\Phi}(\mathbb{R}^n)}}\right) dv \leq 1.
\]
Observe that (by setting \( v := y - x \)), we have
\[
\int_{\mathbb{R}^n} \Phi \left( \frac{|u(y) f(y)|}{u(x)\|f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy = \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y) f(y - x)|}{u(x)\|f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
= \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y + x) f(y)|}{u(x)\|f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
\leq \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y) f(y)|}{u(x)\|f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
= \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y)|}{\|f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
\leq 1.
\]
This shows that \( \|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)} \leq u(x)\|f\|_{L^p_{\Phi}(\mathbb{R}^n)} \).

(2) Let \( f \in L^p_{\Phi}(\mathbb{R}^n) \) and \( f \neq 0 \). Then there exists a constant \( C > 0 \) (depending on \( f \)) such that \( \|f\|_{L^p_{\Phi}(\mathbb{R}^n)} \leq C \). By Lemma 2.4 (1), we have
\[
\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)} \leq C u(x),
\]
for every \( x \in \mathbb{R}^n \). Since \( f(x) \neq 0 \) for every \( x \in \mathbb{R}^n \), we have \( \|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)} > 0 \).

Observe that
\[
\int_{\mathbb{R}^n} \Phi \left( \frac{|u(x) f(v)|}{\sup_{v \in \mathbb{R}^n} u(-v)\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dv \leq \int_{\mathbb{R}^n} \Phi \left( \frac{|u(x) f(v)|}{u(-v)\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dv
\]
\[
\leq \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y + x) f(y)|}{\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
\leq \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y) f(y - x)|}{\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
= \int_{\mathbb{R}^n} \Phi \left( \frac{|u(y)|}{\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right) dy
\]
\[
\leq 1.
\]
This shows that \( \frac{u(x)\|f\|_{L^p_{\Phi}(\mathbb{R}^n)}}{\sup_{v \in \mathbb{R}^n} u(-v)\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \leq \|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)} \).

Choose \( C := \max \left\{ C_1, \frac{\sup_{v \in \mathbb{R}^n} u(-v)\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}}{\|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)}} \right\} \). Hence, we conclude that
\[
\frac{u(x)}{C} \leq \|T_x f\|_{L^p_{\Phi}(\mathbb{R}^n)} \leq C u(x),
\]
as desired. \( \square \)

Now, we present the sufficient and necessary conditions for the inclusion relation between weighted Orlicz spaces \( L^u_{\Phi_1}(\mathbb{R}^n) \) and \( L^w_{\Phi_2}(\mathbb{R}^n) \) with respect to Young functions \( \Phi_1, \Phi_2 \) and weights \( u_1, u_2 \).
Theorem 2.5. Let $\Phi_1, \Phi_2$ be Young functions such that $\Phi_1 \prec \Phi_2$ and $u_1, u_2$ are measurable functions such that $u_i(x + y) \leq u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1, 2$. Then the following statements are equivalent:

1. $u_1 \leq u_2$.
2. $L^{u_1}_{\Phi_2}(\mathbb{R}^n) \subseteq L^{u_2}_{\Phi_1}(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that $\|f\|_{L^{u_1}_{\Phi_2}(\mathbb{R}^n)} \leq C \|f\|_{L^{u_2}_{\Phi_1}(\mathbb{R}^n)}$, for every $f \in L^{u_2}_{\Phi_1}(\mathbb{R}^n)$.

Proof.

Assume that (1) holds. Let $f$ be an element of $L^{u_2}_{\Phi_1}(\mathbb{R}^n)$. Since $\Phi_1 \prec \Phi_2$ and $u_1 \leq u_2$, there exists constants $C_1, C_2 > 0$ such that $\Phi_1(t) \leq \Phi_2(C_1 t)$ for all $t > 0$ and $u_1(x) \leq C_2 u_2(x)$ for every $x \in \mathbb{R}^n$. Using a similar argument as in the proof of Theorem 2.2 we have

$$\|f\|_{L^{u_1}_{\Phi_1}(\mathbb{R}^n)} \leq C_1 \|f\|_{L^{u_1}_{\Phi_2}(\mathbb{R}^n)} \leq C_1 C_2 \|f\|_{L^{u_2}_{\Phi_1}(\mathbb{R}^n)}.$$ 

As before, we have that (2) and (3) are equivalent. It thus remains to show that (3) implies (1). Assume that (3) holds. By Lemma 2.4, we have

$$\frac{u_1(x)}{C} \leq \|T_x f\|_{L^{u_1}_{\Phi_1}(\mathbb{R}^n)} \leq C \|T_x f\|_{L^{u_2}_{\Phi_1}(\mathbb{R}^n)} \leq C u_2(x),$$

for every $x \in \mathbb{R}^n$. So, we obtain $u_1 \leq u_2$. \qed

Note that, for $\Phi_1(x) = \Phi_2(x)$ for every $x \in \mathbb{R}^n$, Theorem 2.5 reduces to Theorem 1.1.

Remark 2.6. It follows from Theorems 2.2 and 2.5 that there cannot be an inclusion relation between $L^{u_1}_{p_1}(\mathbb{R}^n)$ and $L^{u_2}_{p_2}(\mathbb{R}^n)$ for distinct values of $p_1$ and $p_2$. In spite of that, for finite measure set $X$ we can obtain an inclusion relation between $L^{u_1}_{p_1}(X)$ and $L^{u_2}_{p_2}(X)$ as presented in the next section.

3. AN ADDITIONAL CASE

In the following, we will give a sufficient condition for Hölder’s inequality in weighted Orlicz spaces which will be used to obtain an inclusion relation between $L^{u_1}_{p_1}(X)$ and $L^{u_2}_{p_2}(X)$.

Theorem 3.1. (Hölder’s inequality) Let $X$ be a measurable set, $\Phi_1, \Phi_2, \Phi_3$ be Young functions and $u_1, u_2, u_3 : X \to \mathbb{R}$ be measurable functions such that $\Phi_1^{-1}(t) \Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$ for every $t > 0$ and $u_3(x) \leq u_1(x) u_2(x)$ for every $x \in X$. If $f_1 \in L^{u_1}_{\Phi_1}(X)$ and $f_2 \in L^{u_2}_{\Phi_2}(X)$, then $f_1 f_2 \in L^{u_3}_{\Phi_3}(X)$ with

$$\|f_1 f_2\|_{L^{u_3}_{\Phi_3}(X)} \leq 2 \|f_1\|_{L^{u_1}_{\Phi_1}(X)} \|f_2\|_{L^{u_2}_{\Phi_2}(X)}.$$
Proof. Let \( s, t \geq 0 \). Without loss of generality, suppose that \( \Phi_1(s) \leq \Phi_2(t) \). By Lemma 1.2(3), we obtain
\[
st \leq \Phi_1^{-1}(\Phi_1(s))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_1^{-1}(\Phi_2(t))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_3^{-1}(\Phi_2(t)).
\]
Hence \( \Phi_3(st) \leq \Phi_3(\Phi_1^{-1}(\Phi_2(t))) \leq \Phi_2(t) \leq \Phi_2(t) + \Phi_1(s) \). Since \( \Phi \) is a convex function, we have
\[
\int_X \Phi_3 \left( \frac{|u_3(x)|f_1(x)f_2(x)|}{\|f_1\|_{L_{\Phi_1}^u(X)} \|f_2\|_{L_{\Phi_2}^u(X)}} \right) \, dx \leq \frac{1}{2} \int_X \Phi_3 \left( \frac{|u_3(x)|f_1(x)f_2(x)|}{\|f_1\|_{L_{\Phi_1}^u(X)} \|f_2\|_{L_{\Phi_2}^u(X)}} \right) \, dx
\]
\[
\leq \frac{1}{2} \int_X \Phi_3 \left( \frac{|u_1(x)|u_2(x)f_1(x)f_2(x)|}{\|f_1\|_{L_{\Phi_1}^u(X)} \|f_2\|_{L_{\Phi_2}^u(X)}} \right) \, dx.
\]
On the other hand, by Lemma 2.1, we obtain
\[
\int_X \Phi_3 \left( \frac{|u_1(x)|u_2(x)f_1(x)f_2(x)|}{\|f_1\|_{L_{\Phi_1}^u(X)} \|f_2\|_{L_{\Phi_2}^u(X)}} \right) \, dx \leq \int_X \Phi_1 \left( \frac{|u_1(x)|f_1(x)|}{\|f_1\|_{L_{\Phi_1}^u(X)}} \right) \, dx
\]
\[
+ \int_X \Phi_2 \left( \frac{|f_2(x)|}{\|f_2\|_{L_{\Phi_2}^u(X)}} \right) \, dx \leq 2,
\]
whenever \( f_1 \in L_{\Phi_1}^u(X) \) and \( f_2 \in L_{\Phi_2}^u(X) \). By the definition of \( \| \cdot \|_{L_{\Phi_1}^u(X)} \), we have \( \|f_1f_2\|_{L_{\Phi_1}^u(X)} \leq 2\|f_1\|_{L_{\Phi_1}^u(X)}\|f_2\|_{L_{\Phi_2}^u(X)} \), as desired. \( \square \)

Corollary 3.2. Let \( X := B(a, r_0) \subseteq \mathbb{R}^n \) for some \( a \in \mathbb{R}^n \) and \( r_0 > 0 \). If \( \Phi_1, \Phi_2 \) are two Young functions, \( u_1, u_2 : X \to \mathbb{R} \) are measurable functions and there exist a Young function \( \Phi \) and a weight \( 0 < u(x) \leq 1 \) for every \( x \in X \) such that \( \Phi_1^{-1}(t)\Phi^{-1}(t) \leq \Phi_2^{-1}(t) \) for every \( t \geq 0 \) and \( u_1(x) \leq u(x)u_2(x) \) for every \( x \in X \), then
\[
L_{\Phi_1}^u(X) \subseteq L_{\Phi_2}^u(X)
\]
with \( \|f\|_{L_{\Phi_2}^u(X)} \leq \frac{2}{\Phi^{-1}(1/(u(x)/a_{\Phi_1}))} \|f\|_{L_{\Phi_1}^u(X)} \) for \( f \in L_{\Phi_1}^u(X) \).

Proof. Since \( 0 < u(x) \leq 1 \) for every \( x \in X \), we have \( \|f\|_{L_{\Phi_2}^u(X)} \leq \frac{\|f\|_{L_{\Phi_1}^u(X)}}{u} \). Let \( f \in L_{\Phi_1}^u(X) \). By Theorem 3.1 and choosing \( g := \chi_{B(a, r_0)} \), we obtain
\[
\|f\|_{L_{\Phi_2}^u(X)} = \|f\chi_{B(a, r_0)}\|_{L_{\Phi_2}^u(X)}
\]
\[
= \|fg\|_{L_{\Phi_2}^u(X)}
\]
\[
\leq \frac{2}{u} \|f\|_{L_{\Phi_2}^u(X)}
\]
\[
\leq 2 \|f\|_{L_{\Phi_1}^u(X)}
\]
\[
= \frac{1}{\Phi^{-1}(1/(u(x)/a_{\Phi_1}))} \|f\|_{L_{\Phi_1}^u(X)}.
\]

This shows that $L^{u_1}_{\Phi_1}(X) \subseteq L^{u_2}_{\Phi_2}(X)$. \hfill \Box

We shall now discuss the inclusion properties of weighted weak Lebesgue spaces $L^{u_1}_{p_1}(X)$ and $L^{u_2}_{p_2}(X)$ with respect to distinct values of $p_1$ and $p_2$ as well as $u_1$ and $u_2$.

**Corollary 3.3.** Let $X := B(a, r_0)$ for some $a \in \mathbb{R}^n$ and $r_0 > 0$. If $1 \leq p_2 < p_1 < \infty$ and $u_1, u_2 : X \to \mathbb{R}$ are measurable functions such that $u_1(x) \leq u_2(x)$ for every $x \in X$, then

$$L^{u_1}_{p_1}(X) \subseteq L^{u_2}_{p_2}(X).$$

**Proof.** Let $\Phi_1(t) := t^{p_1}, \Phi_2(t) := t^{p_2}$, and $\Phi(t) := t^{\frac{p_1 p_2}{p_1 - p_2}}$ for every $t \geq 0$. Since $1 \leq p_2 < p_1 < \infty$, we have $\frac{p_1 p_2}{p_1 - p_2} > 1$. Thus, $\Phi_1$, $\Phi_2$, and $\Phi$ are Young functions. Now, define $u(x) = \frac{u_1(x)}{u_2(x)}$ for every $x \in X$. Observe that, using the definition of $\Phi^{-1}$ and Lemma 1.2, we have

$$\Phi^{-1}_1(t) = t^{\frac{1}{p_1}}, \Phi^{-1}_2(t) = t^{\frac{1}{p_2}}, \text{ and } \Phi^{-1}(t) = t^{\frac{p_1 p_2}{p_1 - p_2}}.$$

Moreover, $\Phi^{-1}_1(t) \Phi^{-1}(t) = t^{\frac{1}{p_1}} t^{\frac{p_1 p_2}{p_1 - p_2}} = t^{\frac{1}{p_2}} = \Phi^{-1}_2(t)$ and $u_1(x) = \frac{u_1(x)}{u_2(x)} u_2(x) = u(x) u_2(x)$. So it follows from Corollary 3.2 that

$$\|f\|_{L^{u_2}_{p_2}(X)} \leq 2 |B(a, r_0)|^{\frac{p_1 - p_2}{p_1 - p_2}} \|f\|_{L^{u_1}_{p_1}(X)},$$

and therefore we can conclude that $L^{u_1}_{p_1}(X) \subseteq L^{u_2}_{p_2}(X)$. \hfill \Box

4. CONCLUDING REMARKS

We have shown the inclusion properties of weighted Orlicz spaces for distinct Young functions $\Phi_1, \Phi_2$ and weights $u_1, u_2$. The inclusion properties of weighted Orlicz spaces are a generalization of inclusion properties of Orlicz spaces in [7] and inclusion properties of weighted Lebesgue spaces. In the proof of our results, we used the norm of characteristic function on $\mathbb{R}^n$ and estimated the norm of the translation functions in $\mathbb{R}^n$.

Furthermore, from Theorem 2.5 and Lemma 1.1, Theorem 2.8 in [10], we also have the following inclusion relations

$$L^{u_2}_{\Phi_2} \xrightarrow{\text{for } \Phi_1 \prec \Phi_2} L^{u_1}_{\Phi_1}$$

for $\Phi_1 \prec \Phi_2$ and $u_1 \leq u_2$, where the arrows mean ‘contained in’ or ‘embedded into’.

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