DEGREE SUM EXPONENT DISTANCE ENERGY OF SOME GRAPHS

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Abstract. The degree sum exponent distance matrix $M_{\chi_{\text{dist}}}(G)$ of a graph $G$ is a square matrix whose $(i,j)^{th}$ entry is $(d_i + d_j)^{d_{ij}}$ whenever $i \neq j$, otherwise it is zero, where $d_i$ is the degree of the $i^{th}$ vertex of $G$ and $d_{ij} = d(v_i, v_j)$ is distance between $v_i$ and $v_j$. In this paper, we define degree sum exponent distance energy $E_{\chi_{\text{dist}}}(G)$ as sum of absolute eigenvalues of $M_{\chi_{\text{dist}}}(G)$. Also, we obtain some bounds on the degree sum exponent distance energy of some graphs and deduce direct expressions for some graphs.

Key words and Phrases: Eigenvalue, degree sum exponent distance matrix, degree sum exponent distance energy

1. INTRODUCTION

The concept of graph energy was introduced by I.Gutman in 1978[1] having direct correlation with the total $\pi$-electron energy of a molecule in the quantum chemistry as calculated with the Huckel molecular orbital method. Here adjacency matrix of a graph is considered. Later Laplacian energy [2, 4], signless Laplacian energy [3], were introduced. Recently several results on energy related with degree of a vertex and distance in a graph were studied such as distance energy [5], degree sum energy of some graphs [6], degree square sum polynomial of some graphs [8], degree sum energy [9], a survey on energy of graphs [7], complementary distance energy[10], degree sum distance energy [11], degree product distance energy[12], degree exponent energy[13] and degree exponent sum energy[16].

For every pair of vertices in a connected graph there are, degree associated each one of them and in addition there is distance between them (length of the
shortest path). In continuation with this, in order to upgrade, we now introduce concept of degree sum exponent distance energy of connected graph which is slight generalization of degree sum energy since if exponent is made one, it coincides with degree sum energy. The purpose of this paper is to compute the characteristic polynomial, eigenvalues and energy of the new matrix associated with graph, called degree sum exponent distance matrix, and compute bounds for degree sum exponent distance energy and obtain expressions for some standard graphs.

2. Degree Sum Exponent Distance Energy

Let \( G \) be a connected graph of order \( n \) with vertex set \( V(G) = (v_1, v_2, ..., v_n) \). We denote \( d(v_i) \) as the degree of a vertex \( v_i \) which is the number of edges incident on it and \( d_{ij} \) as the distance between two vertices \( v_i \) and \( v_j \), the length of the shortest path joining them. We define degree sum exponent distance matrix of \( G \) as,

\[
M_{\chi_{\text{dist}}}(G) = [\chi_{ij}]
\]

where,

\[
\chi_{ij} = \begin{cases} 
(d(v_i) + d(v_j))^{d_{ij}} & \text{if } i \neq j \\
0 & \text{if } i = j 
\end{cases}
\] (1)

Example: For graph \( G \) given below,

\[
DSED(G) = \begin{pmatrix} 
0 & 5 & 5 & 4 \\
5 & 0 & 4 & 9 \\
5 & 4 & 0 & 9 \\
4 & 9 & 9 & 0 
\end{pmatrix}
\]

Eigenvalues are, \(-11.1616, -4, -3.2007, 18.3623\), and energy is \( E_{\chi_{\text{dist}}}(G) = 36.7246 \).

We note that,

1. \( M_{\chi_{\text{dist}}}(G) \) is real symmetric, so that the eigenvalues of \( M_{\chi_{\text{dist}}}(G) \) are real. If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the eigenvalues of \( M_{\chi_{\text{dist}}}(G) \) then, they can be arranged in a non-increasing order as \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \).
2. \( \sum_{i=1}^{n} \alpha_i = 0 \), since \( \text{trace}[M_{\chi_{\text{dist}}}(G)] = 0 \).
3. The highest exponent term corresponds to \( \text{diam}(G) \).
4. For any \( r \)-regular graph all the entries in the matrix are in powers of \( 2r \).
5. Two non-isomorphic graphs having same order, regularity as well as diameter have same largest eigenvalue \( \alpha_1 \).

We define the degree sum exponent distance energy of a graph \( G \) as,

\[
E_{\chi_{\text{dist}}}(G) = \sum_{i=1}^{n} |\alpha_i|.
\]
3. Bounds on Degree Sum Exponent Distance Energy and Eigenvalues

In this section, we obtain some bounds on degree sum exponent distance energy and largest eigenvalue.

Lemma 3.1. Let $G$ be a graph of order $n$, then we have,
\[ 
\sum_{i=1}^{n} \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i^2 = 2M, \]
where we define, $M = \sum_{i=1,i<j}^{n} ((d_i + d_j)^{d_{ij}})^2$

Lemma 3.2. \[15\] Let $a_1, a_2, ..., a_n$ be non negative numbers. Then,
\[ 
\left( \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \leq \left( \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \leq \left( \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \right)^2
\]

Lemma 3.3. The Cauchy-Schwarz inequality: Let $a_i$ and $b_i$, $1 \leq i \leq n$ be any real numbers, then
\[ 
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)
\]

 Lemma 3.4. \[17\] Let $A$, $B$, $I$ (identity matrix) and $J$ (matrix of all 1's) be square matrices of same order $n$, then block determinant of order $n$,
\[ |AI_n + B(J_n - I_n)| = |A - B|^{n-1} |A + (n - 1)B|
\]

Theorem 3.5. If $\alpha_1$ is the index (largest degree sum exponent distance eigenvalue) of a connected graph $G$ of order $n$, then
\[ \alpha_1 \leq \sqrt{\frac{2M(n-1)}{n}} \]
where $M$ is defined above, with $d_{ij} = d(v_i, v_j)$ the distance between $v_i$ and $v_j$.

Proof. The trace of $M_{\chi_{\text{dist}}}(G)$ being zero we have
\[ \sum_{i=1}^{n} \alpha_i = 0 \text{ i.e, } \sum_{i=2}^{n} \alpha_i = -\alpha_1 \]
Further $\sum_{i=1}^{n} \alpha_i^2 = \text{trace} M_{\chi_{\text{dist}}}(G)^2 = 2M$, where $M$ is as defined above. Using Lemma 3.3, with $a_i = 1$ and $b_i = \alpha_i$ $i = 2, 3, ..., n$ substituting we get,
\[ \left( \sum_{i=2}^{n} \alpha_i \right)^2 \leq (n-1) \sum_{i=2}^{n} \alpha_i^2 \leq (n-1)(2M - \alpha_1^2) \]
Therefore, $(-\alpha_1)^2 \leq (n-1)(2M - \alpha_1^2)$. Simplifying further, the bound for the index $\alpha_1$ follows. \qed

For graph $G$ in Fig 2.1, $\alpha_1 = -11.1616$, $n = 4$ and $M = 244$. We have,
\[ \sqrt{\frac{2M(n-1)}{n}} = 19.1312 \]
Theorem 3.6. If $G$ is connected graph of order $n$ and $M$ is defined above, then
\[\sqrt{2M} \leq E_{\chi_{\text{dist}}}(G) \leq \sqrt{2MN} \]

Proof. With $a_i = 1$ and $b_i = |\alpha_i|$ and using Lemma 3.3 that is,
\[
(\sum_{i=1}^{n} |\alpha_i|)^2 \leq n \sum_{i=1}^{n} (|\alpha_i|)^2.
\]
That is, $E_{\chi_{\text{dist}}}(G)^2 \leq 2nM$.
Hence, $E_{\chi_{\text{dist}}}(G) \leq \sqrt{2MN}$.
Now for the other part,
\[
E_{\chi_{\text{dist}}}(G)^2 = (\sum_{i=1}^{n} |\alpha_i|)^2 \geq \sum_{i=1}^{n} |\alpha_i|^2 = 2M
\]
so that $E_{\chi_{\text{dist}}}(G) \geq \sqrt{2M}$. Combining these two, inequality follows. \(\Box\)

For graph $G$ in Fig 2.1, we have $\sqrt{2M} = 22.09072203$ and $\sqrt{2MN} = 44.1844407$.

Theorem 3.7. If $G$ is any graph of order $n$ and $\Delta$ is the absolute value of the determinant of $\chi_{\text{dist}}(G)$ then,
\[\sqrt{2M + n(n-1)\Delta^2} \leq E_{\chi_{\text{dist}}}(G) \leq \sqrt{2MN}\]
where $M$ is defined as above.

Proof. For lower bound consider,
\[|E_{\chi_{\text{dist}}}(G)|^2 = (\sum_{i=1}^{n} |\alpha_i|)^2 = \sum_{i=1}^{n} (\alpha_i)^2 + 2 \sum_{i<j} |\alpha_i||\alpha_j|\]
Since Arithmetic Mean (AM) $\geq$ Geometric Mean (GM) we have,
\[
\frac{1}{n(n-1)} \sum_{i \neq j} |\alpha_i||\alpha_j| \geq \left(\prod_{i \neq j} |\alpha_i||\alpha_j|\right)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (|\alpha_i|^{2n-2})^{\frac{1}{n(n-1)}} = (\prod_{i=1}^{n} |\alpha_i|^\frac{2}{n}) = \Delta^\frac{2}{n}
\]
Therefore we have, $\prod_{i \neq j} |\alpha_i||\alpha_j| \geq n(n-1)\Delta^\frac{2}{n}$.
Combining we get, $|E_{\chi_{\text{dist}}}(G)|^2 \geq 2M + n(n-1)\Delta^\frac{2}{n}$
ie, $E_{\chi_{\text{dist}}}(G) \geq \sqrt{2M + n(n-1)\Delta^\frac{2}{n}}$ \hspace{1cm} (1)

For upper bound define,
\[
X = \sum_{i=1}^{n} \sum_{j=1}^{n} (|\alpha_i| + |\alpha_j|)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (|\alpha_i|^2 + |\alpha_j|^2) + 2(\sum_{i,j=1, i \neq j} |\alpha_i||\alpha_j|)
\]
\[
= n \sum_{i=1}^{n} (\alpha_i)^2 + n \sum_{j=1}^{n} (\beta_j)^2 - 2(\sum_{i,j=1, i \neq j} |\alpha_i||\alpha_j|)
\]
\[
= 2nM + 2nM - 2|E_{\chi_{\text{dist}}}(G)|^2 = 4nM - 2|E_{\chi_{\text{dist}}}(G)|^2
\]
Since $X \geq 0$ we get $E_{\chi_{\text{dist}}}(G) \leq \sqrt{2MN}$ \hspace{1cm} (2)
Combining lower bound and upper bound, we arrive at the desired result. 

For the graph $G$ in Fig 2.1, $\Delta = 2624$ and $\sqrt{2M + n(n - 1)\Delta^2/n} = 26.20495997$.

**Theorem 3.8.** Let $G$ be a connected $n$ vertex graph and $\Delta$ is the absolute value of the determinant of degree sum exponent distance matrix $\chi_{dist}(G)$, then

$$\sqrt{2M + n(n - 1)\Delta^2/n} \leq E_{\chi_{dist}}(G) \leq \sqrt{2(n - 1)M + n\Delta^2/n}$$

where $M$ is defined as above.

**Proof.** Let $a_i = \alpha_i^2$, $i = 1, 2, \ldots, n$. Then from Lemma 3.1 and Lemma 3.2 we obtain

$$n\left[\frac{1}{n} \sum_{i=1}^{n} \left(\prod_{j=1}^{n} a_j^{1/n}\right)^2\right] \leq n\left[\sum_{i=1}^{n} \alpha_i^2 - (\sum_{i=1}^{n} a_i)^2\right] \leq n(n - 1)\left[\frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 - (\prod_{i=1}^{n} a_i^{1/n})\right]$$

i.e,

$$2M - n\Delta^2/n \leq 2nM - [E_{\chi_{dist}}(G)]^2 \leq 2(n - 1)M - n(n - 1)\Delta^2/n$$

Thus,

$$2M + n(n - 1)\Delta^2/n \leq [E_{\chi_{dist}}(G)]^2 \leq 2(n - 1)M + n\Delta^2/n$$

We get the desired result. 

For the graph $G$ in Fig 2.1, $\sqrt{2(n - 1)M + n\Delta^2/n} = 40.85217223$.

### 4. Degree Sum Exponent Distance Energy of some graphs

**Theorem 4.1.** The degree sum exponent distance energy of $K_n$ is,

$$E_{\chi_{dist}}(K_n) = 4(n - 1)^2.$$ 

**Proof.** The complete graph $K_n$ is of diameter 1 and hence every pair of vertices are at distance 1 so the degree sum exponent distance matrix of $K_n$ is a matrix with zero diagonal and all non diagonal entries $2(n - 1)$ i.e, the degree sum exponent distance matrix of $K_n$ is $2(n - 1)$ times the adjacency matrix of $K_n$. Since the adjacency energy of $K_n$ is $2(n - 1)$, the degree sum exponent distance energy of $K_n$ will be $4(n - 1)^2$.

**Theorem 4.2.** The degree sum exponent distance energy of $CP(n)$ is,

$$E_{\chi_{dist}}(CP(n)) = 32n(n - 1)^2.$$ 

**Proof.** The cocktail party graph $CP(n)$ denotes the $(2n)$-vertex regular graph of degree $(2n - 2)$ (obtained by deleting $n$ independent edges from the complete graph $K_{2n}$). Using Lemma 3.4, where $A = (2n - 2)^2 A(K_2)$, $B = (2n - 2)J_{2 \times 2}$, $J$ is matrix of all 1’s and $A$ is the adjacency matrix. The degree sum exponent distance polynomial of $CP(n)$ is then given by,

$$|\alpha I - M_{\chi_{dist}}(CP(n))| = [\alpha + 16(n - 1)^2]n[\alpha - 8(n - 1)(2n - 3)]^{n-1}[\alpha - 24(n - 1)^2]$$

which gives, $E_{\chi_{dist}}(CP(n)) = 32n(n - 1)^2$. 

For example, in case of CP(4), eigenvalues are $-64(3\text{times}), 48(2\text{times})$ and 96 giving energy, $E_{\chi_{\text{dist}}}(CP(3)) = 384$.  

**Theorem 4.3.** The degree sum exponent distance energy of crown graph $S_n^0$ is, $E_{\chi_{\text{dist}}}(S_n^0) = 16n(n-1)^3$.

**Proof.** The crown graph is the graph obtained by removing a matching from the complete equi-bipartite graph $K_m,n$. So the structure of the degree product distance matrix of $S_n^0$ is, 

$$M_{\chi_{\text{dist}}}(S_n^0) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where $A$ is a matrix of order $n$ with zero diagonal and all non-diagonal entries as $4(n-1)^2$ and $B$ is the matrix of order $n$ with diagonal entry $8(n-1)^3$ and off diagonal entry $2(n-1)$. The eigenvalues of this matrix are given by eigenvalues of $A + B$ and eigenvalues of $A - B$ see[18]. Separately evaluating characteristic polynomials of $A + B$ and $A - B$ and then multiplying we get, degree sum exponent distance polynomial of crown graph, 

$$|\alpha I - M_{\chi_{\text{dist}}}(S_n^0)| = [\alpha + 2(1-n)^2(6n-5)][\alpha - 2(1-n)^2(2n-1)][\alpha + 2(n-1)(4n^2 - 6n + 1)]^{n-1}[\alpha - 2(n-1)(4n^2 - 10n + 5)]^{n-1}.$$ 

Adding all the absolute eigenvalues, we get the theorem. \hfill □

**Lemma 4.4.** [14] If $a,b,c$ and $d$ are real numbers, then the determinant of the form, 

$$|\begin{pmatrix} (\alpha + a)I_{n_1} - aJ_{n_1} - cJ_{n_1 \times n_2} & \alpha I_{n_1 \times n_2} - bJ_{n_2} \\ -dJ_{n_2 \times n_1} & (\alpha + b)I_{n_2} - bJ_{n_2} \end{pmatrix}|$$

of order $n_1 + n_2$ can be expressed in the simplified form as, 

$$(\alpha + a)^{n_1 - 1}(\alpha + b)^{n_2 - 1}([\alpha - (n_1 - 1)a][\alpha - (n_2 - 1)b] - n_1 n_2 cd)$$

**Theorem 4.5.** The degree sum exponent distance energy of the complete bipartite graph $K_{m,n}$ is, $E_{\chi_{\text{dist}}}(K_{m,n}) = 8m^2(m-1) + 8n^2(n-1)$.

**Proof.** In $K_{m,n}$, $m$ vertices have degree $n$ and $n$ vertices have degree $m$. The diameter being 2, the structure of the degree sum exponent distance matrix is, 

$$M_{\chi_{\text{dist}}}(K_{m,n}) = \begin{pmatrix} 4m^2A(K_m) & (m+n)J_{m \times n} \\ (m+n)J_{n \times m} - 4m^2A(K_n) \end{pmatrix}$$

where $J$ is matrix of all 1’s and $A$ is the adjacency matrix. The degree sum exponent distance polynomial is then given by, 

$$|\alpha I - M_{\chi_{\text{dist}}}(K_{m,n})| = \begin{pmatrix} \alpha I_m - 4m^2A(K_m) & -(m+n)J_{m \times n} \\ -(m+n)J_{n \times m} \alpha I_n - 4m^2A(K_n) \end{pmatrix}.$$ 

Using Lemma 4.4 we get the degree sum exponent distance polynomial, 

$$|\alpha I - M_{\chi_{\text{dist}}}(K_{m,n})| = [\alpha + 4m^2]^{m-1}[\alpha + 4m^2]^{n-1}[\alpha^2 - 4(n^2(m-1) + m^2(n-1))\alpha + 16m^2n^2(n-1)(m-
The quadratic equation above has \[4(n^2(m - 1) + m^2(n - 1))^2 > 4 \times 16m^2n^2(n - 1)(m - 1) - (m + n)^2mn\] hence sum of absolute roots is \[4(n^2(m - 1) + m^2(n - 1))\] and on adding all absolute eigenvalues the theorem follows. □

For example, in case of \(K_{3,4}\), eigenvalues are \(-64(2)\), \(36(3)\), \(91\), \(7702\) and \(144.2298\) giving the energy, \(E_{\chi_{\text{dist}}}(K_{3,4}) = 472\).

**Corollary 4.6.** The degree sum exponent distance energy of the star graph \(K_{1,n}\) is,
\[E_{\chi_{\text{dist}}}(K_{1,n}) = 8(n - 1).\]

**Proof.** Put \(m = 1\) in Theorem 4.5. □

**Corollary 4.7.** The degree sum exponent distance energy of the equi-bipartite graph \(K_{n,n}\) is,
\[E_{\chi_{\text{dist}}}(K_{n,n}) = 16n^2(n - 1).\]

**Proof.** Put \(m = n\) in Theorem 4.5. □

**Theorem 4.8.** If \(B_n\) \((n \geq 3)\) is a book graph of order \((n + 2)\) with triangular pages and size \((2n + 1)\), then \(E_{\chi_{\text{dist}}}\) of \(B_n\) is, \(E_{\chi_{\text{dist}}}(B_n) = 36n - 28\)

**Proof.** The book graph \(B_n\) with triangular pages has two sets of vertices, a set with \(n\) vertices of degree 2 and the remaining 2 vertices of degree \((n + 1)\). The structure of the degree sum exponent distance matrix is,
\[
M_{\chi_{\text{dist}}}(B_n) = \begin{bmatrix}
2(n + 1)A(K_2) & (n + 3)J_{2 \times n} \\
(n + 3)J_{n \times 2} & 16A(K_n)
\end{bmatrix}
\]
where \(J\) is matrix of all 1’s and \(A\) is the adjacency matrix. The degree sum exponent distance polynomial is then given by,
\[
|\alpha I - M_{\chi_{\text{dist}}}(B_n)| = \begin{bmatrix}
\alpha I_2 - 2(n + 1)A(K_2) & -(n + 3)J_{2 \times n} \\
-(n + 3)J_{n \times 2} & \alpha I_n - 16A(K_n)
\end{bmatrix}.
\]

Using Lemma 4.4 we get the degree sum exponent distance polynomial,
\[
|\alpha I - M_{\chi_{\text{dist}}}(B_n)| = |\alpha + 16|^{n-1}[\alpha + 2(n + 1)]|\alpha^2 - 2(9n - 7)\alpha + 2(16(n - 1)(n + 1) - n(n + 3)^2)].
\]

The quadratic equation above has \([2(9n - 7)]^2 > 4 \times 2(16(n - 1)(n + 1) - n(n + 3)^2)\) hence sum of absolute roots is \(2(9n - 7)\) and the theorem follows on adding all absolute eigenvalues. □

Let \(K_n - e\) and \(K_n + e\) denote the graph obtained from complete graph \(K_n\) by deleting an edge, adding an edge respectively.
Theorem 4.9.

\[ E_{\chi_{\text{dist}}}(K_n - \epsilon) = \begin{cases} 
44.3606 & \text{if } n = 4 \\
4[(n - 1)(n - 3) + 2(n - 2)^2] & \text{if } n > 4 
\end{cases} \]  

Proof. The graph \( K_n - \epsilon \) is of diameter 2 and has two vertices with distance two and remaining at distance one.

For \( n = 4 \), using Matlab we have \( E_{\chi_{\text{dist}}}(K_4 - \epsilon) = 44.3606 \). The degree sum exponent distance matrix of \( K_n - \epsilon \) has the form,

\[ M_{\chi_{\text{dist}}}(K_n - \epsilon) = \begin{pmatrix} 
0 & 4(n - 2)^2 & 2(n - 3)J_{1 \times n - 2} \\
4(n - 2)^2 & 0 & 2(n - 3)J_{1 \times n - 2} \\
(2n - 3)J_{n - 2 \times 1} & (2n - 3)J_{n - 2 \times 1} & 2(n - 1)A(K_{n - 2}) 
\end{pmatrix}. \]

So that the degree sum exponent distance polynomial of \( K_n - \epsilon \) is given by,

\[ |\alpha I - M_{\chi_{\text{dist}}}(K_n - \epsilon)| = |\alpha + 4(n - 2)^2|[-\alpha + 2(n - 1)]^{n - 1}|\alpha^2 - 2(n - 1)(n - 3) + 2(n - 2)^2\alpha + 2(n - 2)(4(n - 1)(n - 3)(n - 2) - (2n - 3)^2)| \] for \( n > 4 \). Since \( (2(n - 1)(n - 3) + 2(n - 2)^2)^2 > 4 \times 2(n - 2)(4(n - 1)(n - 3)(n - 2) - (2n - 3)^2) \), the sum of absolute roots of the quadratic is \( (2((n - 1)(n - 3) + 2(n - 2)^2)) \). Hence the theorem. \( \square \)

Theorem 4.10. \( E_{\chi_{\text{dist}}}(K_n + \epsilon) = 2(n - 1)(n - 2) + |\alpha_1| + |\alpha_2| + |\alpha_3| \), where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are roots of the equation,

\[ \alpha^3 - 2(n - 1)(n - 2)\alpha^2 - ((2n - 1)^2(n - 1) + n^4(n - 1) + (n + 1)^2)\alpha + 2(n + 1)(n - 1)((n + 1)(n - 2) - n^2(2n - 1)) = 0. \]

Proof. In \( K_n + \epsilon \) there is one vertex with degree \( n \), one vertex with degree 1 and remaining \( n - 1 \) have degree \( n - 1 \). Thus we get the degree sum exponent distance matrix with suitable labeling as,

\[ M_{\chi_{\text{dist}}}(K_n + \epsilon) = \begin{pmatrix} 
(n + 1) & (2n - 1)J_{1 \times n - 1} \\
(2n - 1)J_{n - 1 \times 1} & n^2J_{n - 1 \times 1} \\
0 & 2(n - 1)A(K_{n - 1}) 
\end{pmatrix}. \]

So that the degree sum exponent distance polynomial of \( K_n + \epsilon \) is given by,

\[ |\alpha I - M_{\chi_{\text{dist}}}(K_n + \epsilon)| = |\alpha + 2(n - 1)|^{n - 1}|\alpha^2 - 2(n - 1)(n - 2)\alpha^2 - ((2n - 1)^2(n - 1) + n^4(n - 1) + (n + 1)^2)\alpha + 2(n + 1)(n - 1)((n + 1)(n - 2) - n^2(2n - 1))| \] on extracting eigenvalues and taking the absolute sum, we get the theorem. \( \square \)

For example, in case of \( K_5 + \epsilon \), eigenvalues are \(-40.5244, -8(\text{times}), -3.5991, 68.1235 \) giving the energy as, \( E_{\chi_{\text{dist}}}(K_5 + \epsilon) = 136.247 \).
\textbf{Definition 4.11} (Vertex Coalescence). If $G_1$ and $G_2$ are any two graphs then the graph obtained by gluing $G_1$ and $G_2$ at a point is $v$ called vertex coalescence denoted by $G_1vG_2$.

\textbf{Definition 4.12} (Edge Coalescence). If $G_1$ and $G_2$ are any two graphs then the graph obtained by merging $G_1$ and $G_2$ on an edge $e$ is called edge coalescence denoted by $G_1eG_2$.

Now we consider the degree sum exponent distance energy of vertex coalescence and edge coalescence of complete graphs of same order. Let $K_n$ be a complete graph of order $n$ then the vertex coalescence of $K_n$ with $K_n$ will be denoted by $K_nO_vK_n$ and the edge coalescence by $K_nO_eK_n$.

$K_nO_vK_n$ has $2n-1$ vertices and $2 \times (\binom{n}{2} C_2)$ edges whereas $K_nO_eK_n$ has $2n-2$ vertices and $2 \times (\binom{n}{2} C_2 - 1)$ edges.

\textbf{Lemma 4.13.} [19] Let $a$ and $b$ be two arbitrary constants, $I$ is the identity matrix and $J$ is an $n \times n$ matrix whose all entries $1$'s. If $A = (a - b)I + bJ$ then the characteristic polynomial of $A$ is, $|\lambda I - A| = |\lambda - a + b|^{n-1} |\lambda - a - (n - 1)b|$.

\textbf{Theorem 4.14.} The degree sum exponent distance energy of the vertex coalescence of two complete graphs $K_n$ for $n \geq 3$ is given by,

$$E_{\chi_{\text{dist}}}(K_nO_vK_n) = (n-1)(2n^2 - 5n + 8) + 2(n-1)\sqrt{(2n^2 - 5n + 8)^2 + 18(n - 1)}.$$ 

\textit{Proof.} The graph $K_nO_vK_n$ has two sets of vertices one at a distance 2 from each other and other at 1, being of diameter 2. With suitable labeling the degree sum exponent distance matrix of $K_nO_vK_n$ takes the form,

$$M_{\chi_{\text{dist}}}(K_nO_vK_n) = \begin{bmatrix} 0 & 3(n-1)J_{1 \times n-1} & 3(n-1)J_{1 \times n-1} \\ 3(n-1)J_{n-1 \times 1} & 2(n-1)A(K_{n-1}) & 4(n-1)^2J_{n-1 \times n-1} \\ 3(n-1)J_{n-1 \times 1} & 4(n-1)^2J_{n-1 \times n-1} & 2(n-1)A(K_{n-1}) \end{bmatrix}$$

So that the degree sum exponent distance polynomial of $K_nO_vK_n$

$$|\alpha I - M_{\chi_{\text{dist}}}(K_nO_vK_n)| = \begin{vmatrix} \alpha & -3(n-1)J_{1 \times n-1} & -3(n-1)J_{1 \times n-1} \\ -3(n-1)J_{n-1 \times 1} & \alpha I_{n-1} - 2(n-1)A(K_{n-1}) & -4(n-1)^2J_{n-1 \times n-1} \\ -3(n-1)J_{n-1 \times 1} & -4(n-1)^2J_{n-1 \times n-1} & \alpha I_{n-1} - 2(n-1)A(K_{n-1}) \end{vmatrix}$$

Using Lemma 4.11 we get the degree sum exponent distance polynomial,

$$|\alpha I - M_{\chi_{\text{dist}}}(K_nO_eK_n)| = [\alpha + 2(n-1)(2n-1)^2 - n + 2][\alpha + 2(n-1)]^{2n-4}[\alpha^2 - 2(n-1)(2n^2 - 5n + 8)\alpha - 18(n-1)^3]$$

On extracting eigenvalues and taking the absolute sum, we get the theorem. \hfill \Box

For example, in case of $K_5O_vK_5$, eigenvalues are $-232, -8(6\text{times}), -4.0555, 284.0555$ giving the energy as, $E_{\chi_{\text{dist}}}(K_5O_vK_5) = 568.111$.

\textbf{Theorem 4.15.} The degree sum exponent distance energy of the edge coalescence of two complete graphs $K_n$ for $n \geq 3$ is given by, $E_{\chi_{\text{dist}}}(K_nO_eK_n) = 4(2n-3) + 4(n-1)(2n^2 - 7n + 7) + 8(n-1)(n-3)$.
Degree sum exponent distance energy of some graphs

Proof. The graph $K_n O_r K_n$ has two sets of vertices one at a distance 2 from each other and other at 1, being of diameter 2. There are two vertices of degree $(2n - 3)$ and remaining $(2n - 4)$ of degree $(n - 1)$. With suitable labeling the degree sum exponent distance polynomial of $K_n O_r K_n$ takes the form,

$$M_{\chi_{\text{dist}}}(K_n O_r K_n) = \begin{bmatrix} 2(2n - 3)A(K_2) & (3n - 4)J_{2x2} & (3n - 4)J_{2x2} \\ (3n - 4)J_{n-2x2} & 2(n-1)A(K_{n-2}) & 4(n-1)^2J_{n-2x2} \\ (3n - 4)J_{n-2x2} & 4(n-1)^2J_{n-2x2} & 2(n-1)A(K_{n-2}) \end{bmatrix}$$

So that the degree sum exponent distance polynomial matrix of $K_n O_r K_n$ is given by,

$$|\alpha I - M_{\chi_{\text{dist}}}(K_n O_r K_n)| = \begin{vmatrix} |\alpha I_2 - 2(2n - 3)A(K_2)| & (3n - 4)J_{2x2} & (3n - 4)J_{2x2} \\ (3n - 4)J_{n-2x2} & |\alpha I_{n-2} - 2(n-1)A(K_{n-2})| & 4(n-1)^2J_{n-2x2} \\ (3n - 4)J_{n-2x2} & 4(n-1)^2J_{n-2x2} & |\alpha I_{n-2} - 2(n-1)A(K_{n-2})| \end{vmatrix}$$

So that the degree sum exponent distance polynomial of $K_n O_r K_n$ is given by,

$$|\alpha I - M_{\chi_{\text{dist}}}(K_n O_r K_n)| = |\alpha + 2(2n - 3)\alpha + 2(n-1)(2n^2 - 7n + 7)||\alpha + 2(n-1)|^{2n-6}|\alpha^2 - 2(2n^3 - 7n^2 + 8n - 4)\alpha + 4(2n - 3)(n-1)((2n-1)(n-3) + 2n - 2) - 4(3n - 4)^2(n - 2)|$$

On extracting eigenvalues and taking the absolute sum, we get the following theorem.

For example, in case of $K_3 O_r K_3$, eigenvalues are $-176, -14, -8(4 \times)$, $-6.7839, 215.2161$ giving the energy as, $E_{\chi_{\text{dist}}}(K_3 O_r K_3) = 444.003$.

From Theorem 4.2 and Theorem 4.3 we see that both $S^0_3$ and $CP(3)$ have same $E_{\chi_{\text{dist}}} = 384$, although $CP(3)$ has 12 edges and $S^0_3$ has 6 edges. We call such graphs as degree sum exponent distance equi-energetic.

Definition 4.16. Two non isomorphic graphs on same number of vertices are said to be degree sum exponent distance equi-energetic if they have same degree sum exponent distance energy.

5. Conclusion

We discussed the degree sum exponent distance energy of graphs. Also, we discussed bounds on the energy of degree sum exponent distance energy. There is scope to investigate degree sum exponent distance energy of graphs with higher diameter, trees, unicyclic graphs etc and also to construct degree sum exponent distance equi-energetic graphs.
REFERENCES