SOME IDENTITIES INVOLVING MULTIPLICATIVE (GENERALIZED) \((\alpha, 1)\)-DERIVATIONS IN SEMIPRIME RINGS

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Abstract. Let \(R\) be a semiprime ring, \(I\) a nonzero ideal of \(R\) and \(\alpha\) be an automorphism of \(R\). A map \(F : R \rightarrow R\) is said to be a multiplicative (generalized) \((\alpha, 1)\)-derivation associated with a map \(d : R \rightarrow R\) such that \(F(xy) = F(x)\alpha(y) + xd(y)\), for all \(x, y \in R\). In the present paper, we shall prove that \(R\) contains a nonzero central ideal if any one of the following holds: (i) \(F[x, y] \pm \alpha[x, y] = 0\), (ii) \(F(x \circ y) \pm \alpha(x \circ y) = 0\), (iii) \(F[x, y] = [F(x), y]_{\alpha, 1}\), (iv) \(F[x, y] = (F \circ y)_{\alpha, 1}\), (v) \(F(x \circ y) = [F(x), y]_{\alpha, 1}\) and (vi) \(F(x \circ y) = (F \circ y)_{\alpha, 1}\), for all \(x, y \in I\).

Key words and Phrases: Semiprime rings, Multiplicative (generalized) \((\alpha, 1)\)-derivations, Ideal.

1. INTRODUCTION

Let \(R\) be an associative ring with center \(Z\). For any \(x, y \in R\), the symbol \([x, y]\) stands for the commutator \(xy - yx\) and symbol \(x \circ y\) denotes for the anti-commutator \(xy + yx\). Recall, a ring \(R\) is prime ring if \(xRy = 0\) implies \(x = 0\) or \(y = 0\) and \(R\) is semiprime ring if \(xRx = 0\) implies \(x = 0\). Let \(\alpha\) and \(\beta\) be automorphisms of \(R\). For any \(x, y \in R\), \([x, y]_{\alpha, \beta} = x\alpha(y) - \beta(y)x\) and \((x \circ y)_{\alpha, \beta} = x\alpha(y) + \beta(y)x\). By considering \(\beta = 1\), where 1 is an identity mapping on \(R\), we have \([x, y]_{\alpha, 1} = x\alpha(y) - yx\) and \((x \circ y)_{\alpha, 1} = x\alpha(y) + yx\). An additive mapping \(d : R \rightarrow R\) is called a derivation if \(d(xy) = d(x)y + xd(y)\) holds for all \(x, y \in R\). The concept of a derivation was extended to generalized derivation by Bresar [2]. An additive mapping \(F : R \rightarrow R\) is said to be a generalized derivation if there exists a derivation \(d : R \rightarrow R\) such that \(F(xy) = F(x)y + xd(y)\) for all \(x, y \in R\).
Inspired by the work of Martindale III [11], Daif [5] introduced the concept of multiplicative derivations. Accordingly, a map \( d : R \to R \) is called a multiplicative derivation of \( R \) if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in R \). Of course, these maps are not necessarily additive. Then the complete description of these maps was given by Goldman and Semrl [9]. Further, Daif and Tammam-El-Sayiad [7] extended the notion of multiplicative derivation to multiplicative generalized derivation of \( R \) if \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \), where \( d \) is derivation on \( R \). Recently, the definition of multiplicative generalized derivation was extended to multiplicative (generalized)-derivation by Dhara and Ali [8] as follows: a map \( F : R \to R \) (not necessarily additive) is said to be a multiplicative (generalized)-derivation if \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \), where \( d \) can be any map (not necessarily additive nor a derivation).

Chang [4] introduced the notion of a generalized \((\alpha, \beta)\)-derivation of a ring \( R \) and investigated some properties of such derivations. Let \( \alpha, \beta \) be mappings of \( R \) into itself. An additive mapping \( F : R \to R \) is called a generalized \((\alpha, \beta)\)-derivation of \( R \) such that \( F(xy) = F(x)\alpha(y) + \beta(x)d(y) \) for all \( x, y \in R \) where \( \alpha \) and \( \beta \) are automorphisms on \( R \). A mapping \( F : R \to R \) is said to be a multiplicative (generalized) \((\alpha, \beta)\)-derivation if there exists a map \( d \) on \( R \) such that \( F(xy) = F(x)\alpha(y) + \beta(x)d(y) \) for all \( x, y \in R \). Obviously every generalized \((\alpha, \beta)\)-derivation is a multiplicative (generalized) \((\alpha, \beta)\)-derivation. In 1992, Daif [6], proved a result that if \( R \) is a semiprime ring, \( I \) be a non-zero ideal of \( R \) and \( d \) is a derivation of \( R \) such that \( d([x, y]) = \pm [x, y] \) for all \( x, y \in I \), then \( I \subseteq Z(R) \). Quadri [12] extended the result of Daif by replacing derivation \( d \) with a generalized derivation in a prime ring. Recently, shauliang [10] studied the identities related to generalized \((\alpha, \beta)\) derivation on prime rings. Asma Ali et al. [1] studied the identities related to multiplicative (generalized) \((\alpha, \beta)\)-derivations in semiprime rings. In this line of investigation, in the present paper we shall prove that \( R \) contains a non-zero central ideal if any one of the following holds: (i) \( F[x, y] + \alpha [x, y] = 0 \), (ii) \( F(x \circ y) \pm \alpha (x \circ y) = 0 \), (iii) \( F[x, y] = [F(x), y]_{\alpha, 1} \), (iv) \( F[x, y] = (F(x) \circ y)_{\alpha, 1} \), (v) \( F(x \circ y) = [F(x), y]_{\alpha, 1} \), (vi) \( F(x \circ y) = (F(x) \circ y)_{\alpha, 1} \), for all \( x, y \in I \).

Throughout the present paper, we shall make use of the following basic identities without any specific mention:

1. \( x, yz = y[x, z] + [x, y]z, \)
2. \( xy, z = [x, z]y + x[y, z], \)
3. \( x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z, \)
4. \( xy \circ z = x(y \circ z) - [x, z]y = (x \circ z) y + x[y, z], \)
5. \( xy, z = x[y, z]_{\alpha, 1} + [x, z]y = x[y, \alpha(z)] + [x, z]_{\alpha, 1}y, \)
6. \( xy, z = y[x, z]_{\alpha, 1} + [x, y]_{\alpha, 1} \alpha(z), \)
7. \( (x \circ (yz))_{\alpha, 1} = (x \circ y)_{\alpha, 1} \alpha(z) - y [x, z]_{\alpha, 1} = y(x \circ z)_{\alpha, 1} + [x, y]_{\alpha, 1} \alpha(z), \)
8. \( (xy) \circ z = x(y \circ z)_{\alpha, 1} - [x, z]y = (x \circ z)_{\alpha, 1} y + x[y, \alpha(z)]. \)
2. MAIN RESULTS

In order to prove our main theorems, we shall need the following lemma.

**Lemma 2.1.** ([13, Lemma 2.1]) Let $R$ be a semiprime ring and $I$ is a nonzero two sided ideal of $R$ and $a \in R$ such that $axa = 0$ for all $x \in I$, then $a = 0$.

**Theorem 2.2.** Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $\alpha$ is an automorphism of $R$. Suppose that $F$ is multiplicative (generalized) $(\alpha, 1)$-derivation on $R$ associated with the map $d$ on $R$. If $F [x, y] + \alpha [x, y] = 0$ holds for all $x, y \in I$, then $d$ is commuting on $I$.

**Proof.** By the hypothesis, we have

\[ F [x, y] = 0 \text{ for all } x, y \in I. \]  

(2.1)

Replacing $y$ by $yx$ in (2.1), we obtain that

\[ F ([x, y] x) = 0 \text{ for all } x, y \in I, \]

and so

\[ F ([x, y]) \alpha (x) + [x, y] d (x) = 0 \text{ for all } x, y \in I. \]

Using the hypothesis, we obtain

\[ [x, y] d (x) = 0 \text{ for all } x, y \in I. \]  

(2.2)

Replacing $y$ by $ry$ in (2.2), we get

\[ r [x, y] d (x) + [x, r] yd (x) = 0 \text{ for all } x, y \in I, r \in R. \]

Using (2.2), we obtain

\[ [x, r] yd (x) = 0 \text{ for all } x, y \in I, r \in R. \]  

(2.3)

Replacing $y$ by $yx$ in (2.3), we get

\[ [x, r] yxd (x) = 0 \text{ for all } x, y \in I, r \in R. \]  

(2.4)

Right multiplying (2.3) by $x$, we have

\[ [x, r] yd (x) x = 0 \text{ for all } x, y \in I, r \in R. \]  

(2.5)

Subtracting (2.4) from (2.5), we get

\[ [x, r] y [x, d (x)] = 0 \text{ for all } x, y \in I, r \in R. \]

Replacing $r$ by $d (x)$ in the last equation, we have

\[ [x, d (x)] y [x, d (x)] = 0 \text{ for all } x, y \in I. \]

That is

\[ [x, d (x)] I [x, d (x)] = 0 \text{ for all } x \in I. \]

By lemma 2.1, we conclude that $[x, d (x)] = 0$ for all $x \in I$. Therefore $d$ is commuting on $I$. 
Theorem 2.3. Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $\alpha$ is an automorphism of $R$. Suppose that $F$ is multiplicative (generalized) $(\alpha, 1)$-derivation on $R$ associated with the map $d$. If $F(x \circ y) \pm \alpha(x \circ y) = 0$ holds for all $x, y \in I$, then $d$ is commuting on $I$.

Proof. By the hypothesis, we have
$$F(x \circ y) = 0 \text{ for all } x, y \in I. \quad (2.6)$$
Replacing $y$ by $yx$ in (2.6), we obtain that
$$F((x \circ y)x) \pm \alpha((x \circ y)x) = 0 \text{ for all } x, y \in I,$$
and so
$$F((x \circ y)x) \alpha(x) \pm ((x \circ y)x) \alpha(x) = 0 \text{ for all } x, y \in I.$$
Using the hypothesis, we obtain
$$(x \circ y)d(x) = 0 \text{ for all } x, y \in I. \quad (2.7)$$
Replacing $y$ by $ry$ in (2.7), we find that
$$r(x \circ y)d(x) + [x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R.$$
Using (2.7), we obtain
$$[x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.8)$$
Using the same arguments as used in the proof of Theorem 2.2, we get the required result.

Theorem 2.4. Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $\alpha$ is an automorphism of $R$. Suppose that $F$ is multiplicative (generalized) $(\alpha, 1)$-derivation on $R$ associated with the map $d$. If $F[x, y] = [F(x), y]_{\alpha, 1}$ holds for all $x, y \in I$, then $d$ is commuting on $I$.

Proof. By the hypothesis, we have
$$F[x, y] = [F(x), y]_{\alpha, 1} \text{ for all } x, y \in I. \quad (2.9)$$
Replacing $y$ by $yx$ in (2.9), we obtain that
$$F([x, y]x) = y[F(x), x]_{\alpha, 1} + [F(x), y]_{\alpha, 1} \alpha(x) \text{ for all } x, y \in I,$$
and so
$$F([x, y]) \alpha(x) + [x, y]d(x) = y[F(x), x]_{\alpha, 1} + [F(x), y]_{\alpha, 1} \alpha(x) \text{ for all } x, y \in I.$$
Using the hypothesis, we obtain
$$[x, y]d(x) = y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I. \quad (2.10)$$
Replacing $y$ by $ry$ in (2.10), we find that
$$r[x, y]d(x) + [x, r]yd(x) = ry[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I, r \in R.$$
Using (2.10), we get
$$[x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.11)$$
Using similar argument as used in the proof of Theorem 2.2, we get the required result.

**Theorem 2.5.** Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $\alpha$ is an automorphism of $R$. Suppose that $F$ is multiplicative (generalized) $(\alpha, 1)$-derivation on $R$ associated with the map $d$. If $F[x, y] = (F(x) \circ y)_{\alpha, 1}$ holds for all $x, y \in I$, then $d$ is commuting on $I$.

**Proof.** By the hypothesis, we have

$$F[x, y] = (F(x) \circ y)_{\alpha, 1} \text{ for all } x, y \in I. \hspace{1cm} (2.12)$$

Replacing $y$ by $yx$ in (2.12), we obtain that

$$F([x, y] x) = (F(x) \circ y)_{\alpha, 1} \alpha(x) - y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I$$

and so,

$$F([x, y] \alpha(x) + [x, y] d(x) = (F(x) \circ y)_{\alpha, 1} \alpha(x) - y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I.$$  

Using the hypothesis, we obtain

$$[x, y] d(x) = -y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I. \hspace{1cm} (2.13)$$

Replacing $y$ by $ry$ in (2.13), we get

$$r [x, y] d(x) + [x, r] yd(x) = -ry[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I, r \in R.$$  

Using (2.13), we get

$$[x, r] yd(x) = 0 \text{ for all } x, y \in I, r \in R. \hspace{1cm} (2.14)$$

Arguing in the similar manner as in Theorem 2.2, we get the result.

**Theorem 2.6.** Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $\alpha$ is an automorphism of $R$. Suppose that $F$ is multiplicative (generalized) $(\alpha, 1)$-derivation on $R$ associated with the map $d$. If $F(x \circ y) = [F(x), y]_{\alpha, 1}$ holds for all $x, y \in I$, then $d$ is commuting on $I$.

**Proof.** By the hypothesis, we have

$$F(x \circ y) = [F(x), y]_{\alpha, 1} \text{ for all } x, y \in I. \hspace{1cm} (2.15)$$

Replacing $y$ by $yx$ in (2.15), we obtain that

$$F((x \circ y) x) = y[F(x), x]_{\alpha, 1} + [F(x), y]_{\alpha, 1} \alpha(x) \text{ for all } x, y \in I,$$

and so

$$F((x \circ y)) \alpha(x) + (x \circ y) d(x) = y[F(x), x]_{\alpha, 1} + [F(x), y]_{\alpha, 1} \alpha(x) \text{ for all } x, y \in I.$$  

Using the hypothesis, we obtain

$$(x \circ y) d(x) = y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I. \hspace{1cm} (2.16)$$

Replacing $y$ by $ry$ in (2.16), we get

$$r (x \circ y) d(x) + [x, r] yd(x) = r y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I, r \in R.$$
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Using (2.16), we have

\[ [x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.17) \]

Arguing in the similar manner as in Theorem 2.2, we get the result.

**Theorem 2.7.** Let \( R \) be a semiprime ring, \( I \) a nonzero ideal of \( R \) and \( \alpha \) is an automorphism of \( R \). Suppose that \( F \) is multiplicative (generalized) \((\alpha, 1)\)-derivation on \( R \) associated with the map \( d \). If \( F(x \circ y) = (F(x) \circ y)_{\alpha, 1} \) holds for all \( x, y \in I \), then \( d \) is commuting on \( I \).

**Proof.** By the hypothesis, we have

\[ F(x \circ y) = (F(x) \circ y)_{\alpha, 1} \text{ for all } x, y \in I. \quad (2.18) \]

Replacing \( y \) by \( yx \) in (2.18), we obtain that

\[ F((x \circ y)x) = (F(x) \circ y)_{\alpha, 1} \alpha(x) - y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I, \]

and so

\[ F((x \circ y)\alpha(x) + (x \circ y)d(x) = (F(x) \circ y)_{\alpha, 1} \alpha(x) - y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I. \]

Using the hypothesis, we obtain

\[ (x \circ y)d(x) = -y[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I. \quad (2.19) \]

Replacing \( y \) by \( ry \) in (2.19), we find that

\[ r(x \circ y)d(x) + [x, r]yd(x) = -ry[F(x), x]_{\alpha, 1} \text{ for all } x, y \in I, r \in R. \]

Using (2.19), we have

\[ [x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \quad (2.20) \]

Arguing in the similar manner as in Theorem 2.2, we get the result.

**Corollary 2.8.** Let \( R \) be a semiprime ring. Suppose that \( F, d \) is a multiplicative (generalized) \((\alpha, 1)\)-derivation of \( R \). If any one of the following holds:

\begin{align*}
(i) & \quad F[x, y] = \pm \alpha [x, y] = 0 \\
(ii) & \quad F(x \circ y) = \pm \alpha (x \circ y) = 0 \\
(iii) & \quad F[x, y] = [F(x), y]_{\alpha, 1} \\
(iv) & \quad F[x, y] = (F(x) \circ y)_{\alpha, 1} \\
(v) & \quad F(x \circ y) = [F(x), y]_{\alpha, 1} \\
(vi) & \quad F(x \circ y) = (F(x) \circ y)_{\alpha, 1} \forall x, y \in R
\end{align*}

then \( d \) is commuting on \( R \).
3. Example

In this, we construct an example to the condition (i) of corollary 2.8 so that the semiprimeness condition of the ring is essential.

**Example 1.** Let \( \mathbb{Z} \) be the set of integers and \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \), \( I = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \). Let us define \( F, d, \alpha \) : \( R \to R \) by \( F \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & -b \\ 0 & c \end{pmatrix}, \quad d \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \quad \alpha \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \). It is easy to verify that \( I \) is an ideal on \( R \), \( F \) is multiplicative (generalized) \((\alpha, 1)\)-derivation associated with the map \( d, \alpha \) is an automorphism on \( R \). We see that \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) is nonzero element of \( R \). It implies that \( R \) is not semiprime.

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